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R ROW.

Royal School Series.

THE

STUDENT'S ALGEBRA.

BY

JAMES MACKEAN, F.E.I.S.

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PREFACE.

THIS book has been arranged on the following plan :—

Definitions are deferred till they are about to be used.

Equations and Problems are introduced at the earliest possible stage, in order to interest beginners.

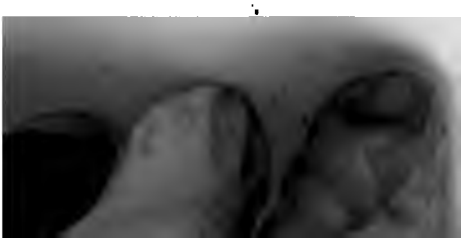
Each rule is accompanied by very full explanations, so that the book may be used for private study as well as in the class-room.

Beginners should read carefully all the explanations, and study the illustrative examples, before attempting the "examples for practice." As many as possible of these last should be then worked out, the harder questions being passed over at first and marked for solution at a later stage.

Most of the examples are original. Some, however, have been taken, with acknowledgment, from Examination Papers and from other works.

JAMES MACKEAN.

December 1880.



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ALGEBRA.

PART FIRST.

CHAPTER I.

DEFINITIONS AND EXPLANATIONS.

1. Algebra is that branch of mathematics in which letters stand for numbers or quantities, and in which symbols and modes of arrangement indicate the relation of these numbers or quantities to one another, and the operations to be performed on them.

2. Numbers may be represented by letters, by combinations of letters, or by combinations of letters and figures, such as, a, m, x; ab, dey; a^2, z^3, b^2cx^3, m^4n^2, $abcx^5y^3z$, etc. These are generally spoken of as quantities.

3. When a letter, or a combination of letters or of letters and figures, is used for a number, twice that letter or combination is used for twice the number, thrice the letter or combination for thrice the number, and so on. Now twice a letter or a combination is indicated by placing 2 in front of it; thrice, by placing 3 in front, etc.

Thus, if a represent 5, $2a$ will represent 10; $3a$ will represent 15; $6a$, 30; $9a$, 45; etc.

In the same way, if a^3b^2c stand for 8, then $2a^3b^2c$ will stand for 16, $5a^3b^2c$ for 40, etc.

The number, therefore, placed in front of a letter or a combination indicates how often that letter or combination is to be taken; that is, it acts as a multiplier.

It is called the coefficient.

In the expression $3a$, 3 is the coefficient; in $4ab^2c$, 4 is the coefficient.

When there is no coefficient expressed, *one* is understood: thus a is the same as $1a$, and x^2y^2 the same as $1x^2y^2$.

4. Sometimes, instead of a number, a letter or letters may stand as the coefficient: thus, in ab, a may be considered the coefficient of b; in abc^2, ab may be taken to be the coefficient of c^2. When letters are thus used, those near the beginning of the alphabet generally form coefficients to those near the end: thus, in ax^2, a^2bxy^2, and $2bcy^3z$, the coefficients of x^2, xy^2, and y^3z are respectively a, a^2b, and $2bc$.

5. It is here to be observed that as long as the letters and the small figures that stand to the right of them remain the same, the quantities are said to be LIKE, whatever the coefficients may be; but if either the letters or the small figures be different, the quantities are said to be UNLIKE.

a, $5a$, and $7a$ are all like quantities.

So also are $12ab^2c$, $7ab^2c$, and $3ab^2c$.

But a, a^2, $4a^2b$, $5ab^2$, and $2axy$ are all unlike quantities.

6. Algebraic quantities are further divided into two kinds, called positive and negative, these being in character directly opposite or contrary to each other. If the posi-

tive signify a sum to be added, then the negative means a sum to be subtracted; if the positive signify a certain amount of money due to a person, then the negative indicates an amount due by a person; if a positive quantity represent motion in a given direction, then the negative stands for motion in a contrary direction.

Positive quantities are marked by placing before them the sign + (read *plus*), and negative ones by placing before them the sign − (read *minus*). When no sign is expressed, *plus* is always understood.

a, $+a^2b$, and mxy are all plus or positive quantities.

$-a$, $-a^2b$, and $-mxy$ are all minus or negative quantities.

ADDITION.

7. The character + (read *plus*) is the sign of addition, and signifies that the quantities between which it is placed are to be added together. Thus, $3+5$ means that 5 is to be added to 3, the sum being 8; and $4+6+8$ that these three numbers are to be added together, the amount being 18.

8. From the different kinds of quantities to be added, algebraic addition naturally divides itself into the following cases:—

I. Addition of Like Quantities.
  (1.) With like signs.
  (2.) With unlike signs.
II. Addition of Unlike Quantities.

9. Let it be required to add $2a$ to $3a$.

Whatever a may be, $2a$ means 2 times a, and $3a$ means 3 times a (Art. 3); their sum must therefore be 5 times a, that is, $5a$.

So also $6ab^2 + 4ab^2$ must amount to $10ab^2$, and $5xyz + 3xyz + xyz$ to $9xyz$.

If it be required to add $-2a$ to $-3a$, we find, as before, that $2a$ and $3a$ amount to $5a$; but as the given quantities are each negative, their sum must also be negative: it is therefore $-5a$.

In the same way, $-6ab^2 - 4ab^2$ amount to $-10ab^2$; and $-7mn - 5mn - 3mn$ to $-15mn$.

Should the quantities to be added be such as $8a - 5a$, that is, a positive and a negative, we find that algebraical addition becomes arithmetical subtraction. $8a$ is the sum of $3a$ and $5a$, and can be written $3a + 5a$, so that $8a - 5a$ becomes $3a + 5a - 5a$. This may be understood to mean that $5a$ is first to be added to $3a$, and then to be taken away from it (Art. 6), the result being evidently the same as if nothing had been either added or taken away. Therefore the sum of $8a - 5a$ is $3a$.

So also $12xy - 7xy$ is equal to $5xy$.

Now let the negative quantity be the greater, as in $7a^2b - 12a^2b$. This may be written $7a^2b - 7a^2b - 5a^2b$. Here the *plus* $7a^2b$ destroys the *minus* $7a^2b$, and $-5a^2b$ is left untouched; it is therefore the algebraic sum of $7a^2b - 12a^2b$.

In like manner, $-4abc$ is the sum of $abc - 5abc$.

If the given sum contains a number of terms, as in $6abx^2 - 7abx^2 + 8abx^2 - 5abx^2$, we may, by one of the methods already illustrated, find the sum of the first two, then of this answer and the third, and so on to the end. Thus, $6abx^2 - 7abx^2$ amounts to $-abx^2$, $-abx^2 + 8abx^2$ to $7abx^2$, and $7abx^2 - 5abx^2$ becomes $2abx^2$, which is the complete answer. But it would evidently amount to the same thing to take, first the sum of the positive terms by themselves, then the sum of the negative ones, and finally the difference between the two results: thus, the sum of

the plus quantities is $14abx^2$; of the minus ones, $-12abx^2$; and the total sum is $14abx^2 - 12abx^2$, or $2abx^2$.

In the same way the sum of $4a^3xy - 6a^3xy - 8a^3xy + 3a^3xy$ is found to be $-7a^3xy$.

10. From the above examples we easily derive the following:—

I. RULES FOR THE ADDITION OF LIKE QUANTITIES.

(1.) When the signs are like, add the coefficients, prefix the proper sign, and affix the common letters.

(2.) When the signs are not all like, add separately the positive and the negative coefficients, find the difference between the two sums, prefix the sign of the greater, and affix the common letters.

11. When the quantities to be added together are unlike, the process can only be indicated, not performed. We cannot actually add, say, $3a$ and $5b$ together, so as to make one quantity, any more than we can add three horses and five cows. We can only represent their sum by the expression $3a + 5b$.

In the same way, the sum of $3a$ and $-5b$ will be represented by $3a - 5b$.

We have therefore the following:—

II. RULE FOR THE ADDITION OF UNLIKE QUANTITIES.

Write the quantities in order, one after the other, and place its own sign before each.

When some of the quantities are like, add them by themselves first.

Illustrative Examples.

(1.) Find the sum of $7x$, $4x$, $9x$, and $5x$.

$7x + 4x + 9x + 5x$ are equal to $25x$.

(2.) Find the sum of $4a - 8a + 6a - a + 3a$.

$4a - 8a + 6a - a + 3a$ amount to $13a - 9a$ or $4a$.

(3.) Find the sum of $6a - 5b - 2a + 3c + 4b$.

$6a - 5b - 2a + 3c + 4b$ are equal to $4a - b + 3c$.

(4.) Find the sum of $3a - 2x - y + z$, $4a + 3y - 2z$, $6x - 4y + 5z$, and $-2a - 4z$.

In questions of this character, it is usual to place like quantities under one another in columns, the sum of each being written below :—

$$\begin{array}{r} 3a - 2x - y + z \\ 4a \quad + 3y - 2z \\ 6x - 4y + 5z \\ -2a \quad - 4z \\ \hline 5a + 4x - 2y. \end{array}$$

EXAMPLES FOR PRACTICE.—I.

(1.)
$$\begin{array}{r} 3a + 2b - c \\ 5a + 4b - 5c \\ a + 7b - 3c \\ 9a + 3b - 5c \\ \underline{6a + b - c} \end{array}$$

(2.)
$$\begin{array}{r} 7a^2 - 5ab + b^2 \\ a^2 - 6ab + 3b^2 \\ 8a^2 - 4ab + 5b^2 \\ 12a^2 - ab + 7b^2 \\ \underline{10a^2 - ab + 9b^2} \end{array}$$

(3.) Add together $6x^2 + 4xy + 5y^2$, $3x^2 + 7xy + 4y^2$, $2x^2 + 5xy + y^2$, $x^2 + 2xy + 9y^2$, and $4x^2 + xy + y^2$.

(4.) Find the sum of $-5x^3 - 2x^2y^2 - 7y^3$, $-4x^3 - 3x^2y^2 - y^3$, $-x^3 - x^2y^2 - y^3$, and $-5x^3 - 7x^2y^2 - 5y^3$.

(5.)
$$\begin{array}{r} 4a^2 - 2b^2 + c^2 \\ -8a^2 - 3b^2 + 4c^2 \\ 3a^2 - 4b^2 - 4c^2 \\ 5a^2 + 2b^2 + 6c^2 \\ \underline{a^2 - 5b^2 + c^2} \end{array}$$

(6.)
$$\begin{array}{r} 7a^2b - 3abc + 5bc^2 \\ -a^2b - 5abc - 2bc^2 \\ 8a^2b + abc - 3bc^2 \\ -5a^2b + 10abc - 4bc^2 \\ \underline{-2a^2b - 3abc + bc^2} \end{array}$$

(7.) Add $4x - 7y + 3z$, $15x + 5y - 8z$, $2x - 5z$, $7x - 3y + 5z$, $-8x + 2y - 6z$, and $4x + 7y + 11z$.

(8.) What is the sum of $12x^3+10x^2-8x+6$, $-7x^3-5x^2+5$, $18x^3-12x^2-12x-10$, $-15x^3+6x^2-10x$, and $11x^3-8x^2-x+2$?

(9.)
$$\begin{array}{rrrrr} 3a+5b & & -6x+4y & & \\ & +7b-7c & +4x-5y & & \\ 4a & +4c & -8x & & -4z \\ -2a & & -5c & +3y & +5z \\ \hline 3a-6b & +8c & +3x-2y & & -6z \end{array}$$

(10.) Add together $3a^2-4a+3x-7ax$, $4a^2+8ax+5x^2-2x$, $7a-5x+3ax-x^2$, and $-4ax-x+6x^2-2x$.

(11.) Find the sum of $7a^4+2a^2b^2-8a^3b+2ab^3+9a^4-5a^2b^2-5b^4+7a^3b+8a^2b^2-4a^4+9b^4-10ab^3$.

(12.) Reduce to its simplest form—

$3x^2-7x+5x^3+7-2x^4-4x^3+2x+3x^4-5x^3-3x^2+x-6$.

12. Literal Coefficients.—When such quantities as ax^2, $2bx^2$, and $-4cx^2$ are to be added, a, $2b$, and $-4c$ are considered the coefficients of x^2 (Art. 4), and, being unlike quantities, their sum is found by Rule II. (Art. 11) to be $a+2b-4c$. They are then enclosed in curved lines, and placed before the x^2 as its total coefficient, thus: $(a+2b-4c)x^2$.

In the same way, the sum of $3abxy^2$, $-5bcxy^2$, and $7acxy^2$ is $(3ab-5bc+7ac)xy^2$.

A similar method is applied to the summation of such quantities as $(5a-3b)x$, $(4b-2c)x$, and $(2c-a)x$. Here the quantities within the curved lines are the coefficients of x, and their sum having been found by the previous rules to be $4a+b$, the result is written $(4a+b)x$.

EXAMPLES FOR PRACTICE.—II.

Find the sum of—

(1.) $4a^2x$, $3abx$, and $-2b^2x$.

(2.) $3my$, $-5ny$, and $-2my$.

(3.) $a^3x^2 - 3a^2x^2 + 3ax^2 - x^2$.

(4.) $(3a - 3b)yz + (2b - 2c)yz + (c - a)yz$.

(5.) $3z^2 + abz^2 + (5 - 2ab)z^2 - 4z^2$.

(6.) $(5a - 2b)x + 3y - 2ax + (a - 3)y + 2bx$.

(7.) $x^3 + a^2x^2 + ax^3 + ax^2 + x^2 + a^2x^3$.

(8.) $(m - n)xy + (2n - p + 2q)xy + (2p - 3)xy + (5 - 3n)xy + (3m - q)xy + 2nxy - 3pxy - qxy$.

(9.) $(a^2 - 2ab + b^2)x + 2ay + (a^2 + 2ab + b^2)x - 2by$.

(10.) $(a^3 - 3a^2b - b^3)x + (3a^3 + 2ab^2 + b^3)y + b^3z$ and $(2a^2b + ab^2)x + (2a^2b - 2a^3)y + (a^3 - a^2b^2)z$.

(11.)

$$\begin{array}{rcrcr}
a^3x & + & (a^2+b^2)x^2 & & -ax^3 \\
b^3x & & -b^2x^2 & + & (a-b)x^3 \\
(c^3-a^3)x & & -a^2x^2 & & +2bx^3 \\
(b^3-2c^3)x & + & (b^2-c^2)x^2 & + & (c-b)x^3 \\
-b^3x & + & (a^2-b^2)x^2 & & +ax^3
\end{array}$$

(12.)

$$\begin{array}{rcrcr}
3x & + & (2a-4)y & + & (3a-b)z \\
(4a-2)x & + & (3b-2c+1)y & + & (2b-a)z \\
-2cx & + & (3c-2a+3)y & + & (3c-2b)z \\
(3b-1)x & & -3by & + & (b-2a-3c)z \\
(a+2c)x & & -cy & + & (4a-b)z
\end{array}$$

13. Substitutions.—When in any set of quantities the numerical values of the letters are known, it may be required to find the value of the whole set, or, as it is called, the expression.

Thus, if the value of a be 6, of b be 4, and of c be 2, what is the value of the expression $5a - 3b + c$?

Here, as a is equal to 6, $5a$ must be equal to 30, and as b is 4, $3b$ must be 12.

The total numerical value will therefore be, $30 - 12 + 2$, that is, $32 - 12$, or 20.

In like manner, the value of $7a + 9b - 13c$, for the same values of a, b, and c, will be $42 + 36 - 26$, or 52.

EXAMPLES FOR PRACTICE.—III.

Find the values of the following expressions when a is equal to 2, b equal to 4, c to 6, d to 8, x to 7, y to 5, and z to 3—

(1.) $7a+5b+3c-4x$.
(2.) $9c-8d-7y+3z$.
(3.) $5a-2b+3c-4y$.
(4.) $15a-13c+11d-9x+7z-5$.
(5.) $12a-8b+4c-d-x+3y-5z$.
(6.) $4a+7b+10c-13d+11x-9y-7z-3$.

14. Equations.—The phrase "is equal to," or "equal to," being in very frequent use, is generally, for the sake of brevity, represented by the sign = (read *equal to*); and when two algebraic expressions are connected by this sign, we have either an identity or an equation: an identity when the one side may be derived from the other by the operations of addition, subtraction, multiplication, etc.; an equation, when the one side cannot be so derived from the other.

$5x+7x-3x=9x$ is an identity.

$5x+7x-3x=18$, and $6x^2-3=4x^2+5$, are equations.

Observe that an identity remains true whatever values be given to the letter or letters it contains, while an equation is satisfied only by the substitution of some particular number or numbers.

An equation may therefore be described as an equality between two or more quantities, some of which are of known value, some unknown.

15. A quantity whose value is not known is spoken of as an unknown quantity; and the finding the value or values of the unknown quantity or quantities in an equa-

tion constitutes what is called the solution of the equation.

These values when found are termed the roots of the equation.

16. Unknown quantities are generally represented by the last letters of the alphabet; and when these appear by themselves, or with coefficients only, and not in such combinations as x^2, y^3, y^2z, etc., the equation is said to be simple.

$5x + 7x - 3x = 18$ is a simple equation, while $6x^2 - 3 = 4x^2 + 5$ is not.

Simple equations are also called equations of the first degree, or of one dimension.

17. Solution of Simple Equations containing one Unknown Quantity, and involving no algebraic operation higher than addition:—

Find the value of x when $2x = 6$.

Here it is plain that if two times x equal *six*, x itself must equal *three*, the half of *six*; or,

$$x = \tfrac{6}{2} = 3.$$

Again, if $7x = 16$, then x must equal the seventh part of sixteen; or,

$$x = \tfrac{1}{7}\tfrac{6} = 2\tfrac{2}{7}.$$

From this it appears that an equation of the above form is solved by dividing the known quantity by the coefficient of the unknown one.

Given $3x + 9x - 5x = 21$ to find the value of x.

By Art. 10, $7x = 21$

$$\therefore\ x = \tfrac{2}{7}\tfrac{1} = 3.$$

18. The sign $\therefore$ is used to represent *therefore*, and $\because$ to represent *because*.

EXAMPLES FOR PRACTICE.—IV.

Find the value of the unknown quantity in the following equations:—

(1.) $7x + 3x + 5x + 6x = 84$.
(2.) $13x - 7x - 4x + 5x = 42 - 16 + 37$.
(3.) $17x - 12x + 8x - 5x = 24 - 16 + 4$.
(4.) $4x + 7x + 5x - 6x = 56 - 19 - 7$.
(5.) $-9y + 13y - 17y + 21y = 104$.
(6.) $49y - 29y - 19y + 23y = 888$.

19. Problems producing Simple Equations.—In applying algebra to the solution of problems, we put a letter—say x—to represent the quantity which we desire to find. The value of this quantity, of course, is not known, but its relation to some given quantity or quantities must be known; and the first part of the process of solution consists in setting down this relation in algebraic language. This leads us to an equation; and the finding of the value of its unknown quantity completes the solution of the problem.

Illustrative Examples.

(1.) What number added to twice itself makes 105?

There is here stated a relation between the unknown number and the given one, which enables us to form an equation.

Let x = the required number.
Then $2x$ = twice the number.
And $2x + x$ = the number added to twice itself.

But by the question—

105 = the number added to twice itself;

that is, there are two quantities each equal to the same quantity.

Now, it may be taken as an "admitted truth," or axiom, that "Things which are equal to the same thing are equal to one another." We have, therefore, the equation—

$$2x + x = 105$$

By addition $3x = 105$

And $\therefore$ (Art. 17) $x = 35$ = the required number.

Proof, $35 + 2 \times 35 = 35 + 70 = 105$.

(2.) A parcel contained 60 lbs. of tea; there were three times as many lbs. at 2s. 8d. as there were at 3s., and twice as many at 2s. 4d. as at 2s. 8d. How many lbs. were there at each price?

There are here three quantities to be found, but they are so connected with one another that if any one of them be known, the others can immediately be ascertained. It is therefore only necessary to consider this question as involving one unknown quantity, which may be the number of lbs. at 3s., at 2s. 8d., or at 2s. 4d. It will be most convenient to let it be the smallest number.

Let x = the number of lbs. at 3s.

Then $3x$ = the number of lbs. at 2s. 8d.

And $6x$ = the number of lbs. at 2s. 4d.

$\therefore$ $x + 3x + 6x$ = the total number of lbs. in the parcel.

But 60 = the total number of lbs. in the parcel.

By the axiom stated above we have therefore—

$$x + 3x + 6x = 60.$$

Or, $10x = 60$.

$\therefore$ $x = 6$ = number of lbs. at 3s.

$3x = 18$ = number of lbs. at 2s. 8d.

$6x = 36$ = number of lbs. at 2s. 4d.

Proof, $6 + 18 + 36 = 60$.

(3.) Two persons divide £47, 3s. between them in such a manner that for every pound the one receives, the other receives a guinea. What does each receive?

Let x = the number of shares each gets.
Then $20x$ = the number of shillings the 1st gets.
And $21x$ = the number of shillings the 2nd gets.
$\therefore 20x + 21x$ = the total sum in shillings.
But 943 = the total sum in shillings.
$\therefore 20x + 21x = 943.$
Or, $41x = 943.$
$\therefore x = 23.$
$\therefore 20 \times 23 = 460\text{s.} = £23$ = share of 1st.
And $21 \times 23 = 483\text{s.} = £24,\ 3\text{s.}$ = share of 2nd.

EXAMPLES FOR PRACTICE.—V.

(1.) Three boys have among them 72 marbles; the second has twice as many as the first, and the third thrice as many as the first. What number has each?

(2.) A Maypole having been snapped by the wind, it was found that the part broken off was four times the length of the piece left standing. The original height having been 30 feet, it is required to find the lengths of the parts.

(3.) A father is five times as old as his son, and their united ages amount to 36 years. What is the age of each?

(4.) Divide the number 96 into two parts, so that the one may be seven times the other.

(5.) A purse contains £52, 18s., made up of an equal number of guineas, sovereigns, and crowns. How many pieces are there of each kind?

(6.) A dairyman sold 40 quarts of a mixture of milk and water, there being one pint of water to two quarts of milk. How much water had he added?

(7.) £1 was divided among A, B, and C in such a way that for every fivepence A got, B received fourpence, and C threepence. How much did each get?

Let x = number of shares each got.

(8.) A railway train starts from Edinburgh for London at the rate of 21 miles per hour, while another starts at the same time from London for Edinburgh at the rate of 19 miles an hour. The distance being 400 miles, when will they meet, and how far from London?

(9.) Divide £360 among A, B, and C, so that A may have three times as much as C, and B half as much as A and C together.

(10.) The fares on a railway being for third class one penny per mile, second class three-halfpence per mile, and first class twopence per mile, how far can three passengers, one in each class, be carried at the same time for 18s. 9d.?

(11.) At a sale of farm stock a lot, consisting of horses, oxen, and sheep, was bought for £500. There was found to be the same number of horses as of oxen, and twice as many sheep as of either. Now a horse cost £35, an ox £17, 10s., and a sheep £5. How many were there of each?

(12.) In a certain factory a woman earns twice as much as a boy, and a man thrice as much as a woman. There are 50 men, 135 women, and 12 boys, and it takes £101, 17s. weekly to pay their wages. What is a man's wage per week?

CHAPTER II.

SUBTRACTION.

20. Subtraction is indicated by the sign $-$ (read *minus*) being placed before the quantity to be subtracted. Thus, $7-3$ means that 3 is to be subtracted from 7; so, $6a-4a$ means that $4a$ is to be taken from $6a$; and $6a-(-4a)$ means that $-4a$ is to be taken from $6a$.

Sometimes the sign $\sim$ (read *difference*) is placed between two quantities to show that the smaller is to be taken from the greater.

21. The subtraction of $4a$ from $6a$ presents no difficulty; we easily perceive that the remainder must be $2a$; but the difference between $6a$ and $-4a$ is not so readily seen.

It will be at once admitted that if we add any number to another, and then take it away again, no change will be made on the original number. Thus,

$$7=7+5-5, \text{ and } 8x=8x+3x-3x.$$

So, if $4a$ be first added to $6a$, and then taken away, the $6a$ will remain unchanged, or

$$6a=6a+4a-4a.$$

If now from this expression for $6a$ we take away or sub-

tract the $-4a$, the remainder will plainly be $6a+4a$; that is,

$$6a-(-4a)=6a+4a, \text{ or } 10a.$$

In the same way, $12x-(-5x)=12x+5x=17x$, and $-8a^2b^2-(-3a^2b^2)=-8a^2b^2+3a^2b^2=-5a^2b^2$ (Art. 10).

From these examples it appears that, to subtract a plus quantity, we write it with a minus sign; and to subtract a minus quantity, we write it with a plus sign.

22. This may be shown more generally.

Let it be required to subtract $y-z$ from x.

$$\begin{array}{r} x \\ \underline{y-z} \\ x-y+z. \end{array}$$

First subtract y from x. As the quantities are unlike, this can only be done by indicating it; that is, by placing the minus sign before the quantity to be subtracted. The result is $x-y$; but the quantity to be subtracted is less than y by the amount of z: we have therefore, in taking away the whole of y, taken away z too much, and must add z to $x-y$ in order to obtain the true remainder. This gives us $x-y+z$ as the difference between x and $y-z$, in which we observe that the sign of x (which represents the minuend) remains unchanged, while the signs of y and z (the quantities to be subtracted) have both been reversed.

23. From this we have the following:—

RULE OF SUBTRACTION.

Change the signs of the quantities to be subtracted, and proceed as in addition.

Illustrative Examples.

(1.) From $3a^2-5ax-x^2$ take $2a^2-3ax-4x^2$.

Change the signs of the terms to be subtracted, and place them as in addition :—

$$\begin{array}{r} 3a^2 - 5ax - x^2 \\ -2a^2 + 3ax + 4x^2 \\ \hline a^2 - 2ax + 3x^2 \end{array}$$

Here, the $+2a^2$ becomes $-2a^2$, and $-2a^2$ being added to $+3a^2$ gives $+a^2$; $-3ax$ becomes $+3ax$, which, being added to $-5ax$, yields $-2ax$; $-4x^2$ is changed into $+4x^2$, and the sum of $-x^2$ and $+4x^2$ is $+3x^2$. The required remainder is, therefore, $a^2 - 2ax + 3x^2$.

(2.) Take $4a - 5b + 6c - 3y$ from $7a + 5b + 3c - x$.

$$\begin{array}{rl} 7a + 5b + 3c - x & \\ -4a + 5b - 6c + 3y & \text{(Signs changed.)} \\ \hline 3a + 10b - 3c - x + 3y. & \text{(By addition.)} \end{array}$$

The beginner may set down the quantities with the signs changed, as has been done in these two examples; but as soon as possible he ought to acquire the power of making the change mentally, the quantities being set down with their proper signs.

(3.) Subtract $3x^2 - 5xy + 7y^2$ from $-2x^2 - 5xy + 6y^2$.

$$\begin{array}{r} -2x^2 - 5xy + 6y^2 \\ 3x^2 - 5xy + 7y^2 \\ \hline -5x^2 \qquad - y^2 \end{array}$$

24. Literal Coefficients. — When the coefficients are literal, the operation is performed in the same manner, the coefficients being collected and enclosed within curved lines, as in addition (Art. 12).

Illustrative Examples.

(1.) Subtract $3ax$ from $2bx$.

The coefficient of x in the minuend being $2b$, and in the

subtrahend $3a$, the coefficient of the remainder is $2b - 3a$, and the complete answer is therefore $(2b - 3a)x$.

(2.) From

$(5a^2 - 2ab + b^2 + 4c^2)x^2$ take $(4a^2 - 2ab + b^2 + 4ac)x^2$.

$$\begin{array}{r} 5a^2 - 2ab + b^2 + 4c^2 \\ \underline{4a^2 - 2ab + b^2 + 4ac} \\ a^2 - 4ac + 4c^2 \end{array}$$

$$\therefore \textit{Ans.} = (a^2 - 4ac + 4c^2)x^2.$$

(3.) Subtract $ax - 2by - 3cx$ from $3ax - by + 2cy$.

$$\begin{aligned} \textit{Ans.} &= 3ax - by + 2cy - ax + 2by + 3cx \text{ (Art. 23)} \\ &= 2ax + 3cx + by + 2cy \text{ (Art. 10)} \\ &= (2a + 3c)x + (b + 2c)y \text{ (Art. 12).} \end{aligned}$$

(4.) Collect $(3a + b)x + (a - 3c)y + (b - c)x$
$- (2a - 3c)y + (a - 3b)y - (3a - 2c)x$.

$$\begin{aligned} \textit{Ans.} &= (3a + b)x + (b - c)x - (3a - 2c)x \\ &\quad + (a - 3c)y + (a - 3b)y - (2a - 3c)y \\ &= (3a + 2b - c)x - (3a - 2c)x \text{ (Art. 12)} \\ &\quad + (2a - 3b - 3c)y - (2a - 3c)y \\ &= (2b + c)x - 3by \text{ (Art. 24).} \end{aligned}$$

EXAMPLES FOR PRACTICE.—VI.

(1.) Subtract $4axy$ from $7axy$.
(2.) Subtract $-6a^2$ from $5a^2$.
(3.) Subtract $-7ab$ from $-4ab$.
(4.) Subtract $-3xy^2$ from $-8xy^2$.
(5.) Subtract $5abc$ from $-2abc$.
(6.) Subtract $12am$ from $4am$.
(7.) Subtract $3b$ from $5a$.
(8.) Subtract $-7abc$ from $9a^3$.
(9.) Subtract $2a^2 - 3b^2$ from $5a^2 - 4b^2$.
(10.) Subtract $5xy - 6y^2$ from $3xy + 2y^2$.
(11.) Subtract $2a^2 - 3ab - b^2$ from $5a^2 - 3ab - 3b^2$.

(12.) Subtract $3a - 4b + 5c$ from $7a - 2b + c$.

(13.) Subtract $2a^2 - 3b^2 + c^2$ from $2x^2 - 3y^2 + c^2$.

(14.) Take $7ax - 9by - 11cz$ from $4ax - 6by - 8cz$.

(15.) Take $x^3 + 5x^2 - 7x + 6$ from $4x^4 + 3x^3 - 3x^2 - 5x$.

(16.) Subtract $a^2 - 2ab + b^2$ from $a^2 + b^2$, and from the remainder take $2ab - c^2 - d^2$.

(17.) Take the sum of $a^3 - a^2 - a - b^3$ and $a^2b - ab^2$ from $a^3 - 3a^2b + 3ab^2 - b^3$.

(18.) Find the difference between $a - b + c - d$ and $a - y + c - z$.

(19.) From $4a^2x^2$ take $5bcx^2$.

(20.) Take $3cxy$ from $3axy + 2bxy$.

(21.) Subtract $(2mn - m^2)y^2$ from $(4mn - 3n^2)y^2$.

(22.) Take $dex - efy + fgz$ from $abx + bcy + cdz$.

(23.) From $5a^2x - 2y - 3x$ subtract

$$(a^2 - b^2)x - (2b^2 - 1 - 4a^2)y.$$

(24.) From the sum of $(a^2 + 2ab + b^2)x^2$ and $(a^2 - 2ab + b^2)x^2$ take their difference.

25. Brackets.—When an expression containing two or more terms is to be treated as one, it is frequently enclosed in brackets.

Brackets are of various forms, as—

$$(\quad) \qquad \{ \quad \} \qquad [\quad] \qquad \left(\quad \right) \text{ etc.}$$

A line drawn above or below a number of terms is sometimes used for the same purpose, and is called a *vinculum*—as in $\overline{a + b + c}$, and $\frac{3x - y}{4a}$

26. $(3a + 4b - 2c) + (5a - 3b + 5c) + (2a - 2b - c)$.

This expression simply means that the quantities in the second and third pair of brackets are to be added to those

in the first. Therefore, placing them below one another, as in addition, we have—

$$\begin{array}{r} 3a+4b-2c \\ 5a-3b+5c \\ 2a-2b-\ c \\ \hline 10a-\ b+2c \end{array}$$

But it is plain that addition may be performed horizontally as well as vertically, and that if we remove the brackets and write the quantities in one line, the result, when the coefficients of like quantities are collected, must be the same—

$$3a+4b-2c+5a-3b+5c+2a-2b-c=10a-b+2c.$$

27. $(3x-4y+5z)-(5x-3y-z)$.

This means that the quantities within the second pair of brackets are to be taken from those in the first pair; and performing the subtraction in the usual way—

$$\begin{array}{r} 3x-4y+5z \\ 5x-3y-z \\ \hline -2x-\ y+6z \end{array}$$

we have

But it is known (Art. 23) that subtraction is changed into addition by altering the signs of all the quantities to be subtracted.

If, therefore, we write the quantities in one line, taking care, in removing the brackets, to change the signs of the quantities to be subtracted (that is, of those within the brackets preceded by $-$), we shall simply have to collect the coefficients of like quantities, as in addition.

The above sum then becomes—

$$3x-4y+5z-5x+3y+z\ =\ -2x-y+6z.$$

So also—

$$\begin{aligned}(7a^2-5b^2)-(5a^2-4ab+3b^2) &= 7a^2-5b^2-5a^2+4ab-3b^2 \\ &= 2a^2+4ab-8b^2.\end{aligned}$$

28. When an expression in brackets forms part of another also enclosed in brackets, it is said to be within *double brackets*—as in $2x^3 - \{3x^2 - (2x - 1)\}$.

In this case either of the pairs may be removed before the other; thus, removing the outer ones first—

$2x^3 - \{3x^2 - (2x - 1)\} = 2x^3 - 3x^2 + (2x - 1) = 2x^3 - 3x^2 + 2x - 1$;

or, beginning with the inner ones—

$2x^3 - \{3x^2 - (2x - 1)\} = 2x^3 - \{3x^2 - 2x + 1\} = 2x^3 - 3x^2 + 2x - 1$.

29. When several pairs of brackets are employed, as in

$$4x - [3x + 7y - \{3y + 5z - (3z - \overline{2x - y})\}],$$

they may be removed in any order, but it is generally preferable to begin with the outer pair.

A little practice will soon enable the student to decide upon the order to be adopted in any particular case.

$$\begin{aligned} & 4x - [3x + 7y - \{3y + 5z - (3z - \overline{2x - y})\}] \\ = {} & 4x - 3x - 7y + \{3y + 5z - (3z - \overline{2x - y})\} \\ = {} & x - 7y + 3y + 5z - (3z - \overline{2x - y}) \\ = {} & x - 4y + 5z - 3z + \overline{2x - y} \\ = {} & x - 4y + 2z + 2x - y \\ = {} & 3x - 5y + 2z \end{aligned}$$

The writing down of this work may be shortened by leaving out those brackets that are preceded by $+$; thus,

$$\begin{aligned} & 4x - [3x + 7y - \{3y + 5z - (3z - \overline{2x - y})\}] \\ = {} & 4x - 3x - 7y + 3y + 5z - (3z - \overline{2x - y}) \\ = {} & x - 4y + 5z - 3z + 2x - y \\ = {} & 3x - 5y + 2z \end{aligned}$$

EXAMPLES FOR PRACTICE.—VII.

Remove the brackets from the following:—

(1.) $8a - (7x + 3a) + (4x - 2a)$.

(2.) $4x^2 - \{(2x - 3x^2) - (x - 5x^2)\}$.

(3.) $\{(8a^2 + 5b^2) - (3ab + c^2)\} - \{(4a^2 - 3ab) - (3b^2 + c^2)\}$.

(4.) $7x^3 - [6x^3 - \{5x - (3 + x - 2x^2 + 4x^3)\}]$.

(5.) $2a^2 - \{3a - (4 - 5a) - 4a^2\} - (2a^2 - 8a + 2)$.

(6.) $2a - [2a + b - \{2a + 2b - (2a + 3b - \overline{2a + 4b})\}]$.

(7.) $1 - [1 - \{1 - (1 - \overline{1 - 1} - 1) - 1\} - 1]$.

(8.) $x^3 - y^3 - (3x^2y - 3xy^2) - \{x^3 + y^3 - (3x^2y - 3xy^2)\}$.

(9.)

$\{a^3 + b^3 - (a^2b + ab^2)\} - [(a^3 - b^3 + 2ab) - \{a^2b - (ab^2 - 2ab)\}]$.

(10.) $1 - [2x - \{3x^2 - (4x^3 - \overline{5x^4 + 6x^3} + 5x^2) + 4x\} + 3]$.

(11.)

$3a - [4b - \{2a - (4c - \overline{2c - b} - 3b) + 4b - (a - 3b - 2c)\}]$.

(12.)

$x^2 - \{y^2 - (z^2 - z) - y\} - x - [\{y - (x + z)\} - \{y^2 - (x^2 + z^2)\}]$.

30. After the student has wrought these examples by the method indicated above, he may go through them again, removing the whole of the brackets in each example at once. This may be done by observing carefully and remembering the effect of the sign that precedes each pair.

Resolve $a^3 - \underset{1}{[}(a^2 - 1) - \underset{2}{\{}a + 1 - \underset{3}{(}a^3 - \overline{a^2 - 1}\underset{4}{)} + a^2\}]$.

Observe the effect of the signs marked 1, 2, 3, 4, and write down the result.

$$a^3 - a^2 + 1 + a + 1 - a^3 + a^2 - 1 + a^2 = a^2 + a + 1.$$

EXAMPLES FOR PRACTICE.—VIII.

Substitutions.—When $x = 5$, $y = 7$, $z = 9$, $a = 6$, $b = 4$, $c = 2$, find the value of the following:—

(1.) $4a - 6b + 2c$.

(2.) $(a + x) - (b + y) + (c + z)$.

(3.) $5x - \{7y - (9z - 2x)\}$.

(4.) $4a - [4b - \{4c - (3x - \overline{2y - z})\}]$.

(5.) $13a - [7x - \{8y - (3z - c) - 12b\} + 13c]$.

(6.) $a - 1 - [3b - 2 - \{c - 3 - (x - 5 - \overline{y - z})\} - a]$.

31. It is frequently necessary to place within brackets terms that appear singly. From what has been said in Arts. 26 and 27, it follows that when a positive sign is to stand before the brackets, no change in the signs of the terms is required; but when a negative sign is to stand before them, the sign of each single term must be reversed.

$3a+2b-4c$ placed within brackets preceded by + becomes $+(3a+2b-4c)$, or $(3a+2b-4c)$.

The same placed within brackets preceded by − becomes $-(-3a-2b+4c)$, or $-(4c-3a-2b)$.

In placing quantities within brackets, it is almost always required to put before each set the same sign as its first term had before being put in brackets.

Paying attention to this requirement, arrange $a^6-5a^5-3a^4-2a^3-4a^2+5a$ in sets of two—

$$(a^6-5a^5)-(3a^4+2a^3)-(4a^2-5a).$$

Do the same in sets of three—

$$(a^6-5a^5-3a^4)-(2a^3+4a^2-5a).$$

In the last question, enclose the second and third terms of each set within double brackets—

$$\{a^6-(5a^5+3a^4)\}-\{2a^3+(4a^2-5a)\}.$$

EXAMPLES FOR PRACTICE.—IX.

(1.) Place $3a-5b+4c$ in brackets preceded by a positive sign.

(2.) Place $-4a^2+2ab-b^2$ in brackets preceded by a positive sign.

(3.) Place $-ax-by+cz$ in brackets preceded by a negative sign.

(4.) Place a^3-a^2+2a-2 in brackets preceded by a negative sign.

Place the following, first, in sets of two, secondly, in sets of three :—

(5.) (6.) $a+b-c-d+e-f$.

(7.) (8.) $3a+4b+3c-4x-3y-4z$.

Arrange the following in brackets in sets of three, with the second and third of each set within an inner pair :—

(9.) $5a^3-3a^2-2a-4b^3+2b^2-b$.

(10.) $x^3+2x^2-3x-y^3+2y^2+3y$.

(11.) (12.) Enclose the answers in (9.) and (10.) within another pair of brackets preceded by *minus*.

32. Equations involving no higher algebraic process than subtraction.—In the equations already given (Arts. 17 and 19), the term or terms containing the unknown quantity alone appeared on the first side, and the numbers alone on the second side. Now, the unknown quantity and the numbers may appear on *either* or on *both* sides, and the first part of the process of solving consists in arranging the terms so that those containing the unknown quantity shall be all on the one side, while all the numbers are on the other.

Illustrative Examples.

(1.) Suppose $8x+11=51$.

Let 11 be subtracted from both sides.

Then $8x+11-11=51-11$.

But $11-11=0$.

$\therefore 8x=51-11$.

Observe that the number 11, which was given on the first side, now appears on the second side with its sign changed.

The solution is completed as in Art. 17.

$$8x=40.$$

$$\therefore x=5.$$

(2.) Let $11x = 40 + 3x$.

Subtract $3x$ from both sides.

Then $11x - 3x = 40 + 3x - 3x$.

Or $11x - 3x = 40$, for $3x - 3x = 0$.

$8x = 40$, and $\therefore\ x = 5$.

In the second last line, $3x$ given on the second side appears on the first side with its sign changed.

(3.) $7x + 12 + 5x = 27 + 4x + 25$.

1st, Collect like terms—

$$12x + 12 = 52 + 4x.$$

2nd, Take 12 and $4x$ from each side.

Then $12x - 4x + 12 - 12 = 52 - 12 + 4x - 4x$.

Or $12x - 4x = 52 - 12$

$8x = 40$, and $\therefore\ x = 5$.

Here again in the second last line a number given on the first side with the sign (+) appears on the second with the sign (−), and the quantity $4x$ given (+) on the second side becomes (−) on the first.

In changing their sides they have changed their signs.

(4.) Given $6x - 25 = 15 - 2x$.

In order to bring all the unknown quantities to the first side and the numbers to the second, it will here be necessary to *add* $2x$ and 25 to both sides.

Then $6x - 25 + 2x + 25 = 15 - 2x + 2x + 25$

Or $6x + 2x = 15 + 25$.

$\therefore\ 8x = 40$, and $x = 5$.

Once more the quantities whose sides have been changed have changed their signs.

33. From this we have the following

RULE.—Arrange all the terms containing the unknown quantity on one side (generally the first), and all those

containing numbers on the other, changing, at same time, the signs of those whose sides are changed. Complete the solution as in Article 17.

(5.) $12x-23+7x+5 = 4x-8-9x+38$
$12x+7x-4x+9x = 23-5-8+38$
$28x-4x = 61-13$
$24x = 48, \quad \therefore x = 2.$

(6.) $7x-13-12x+5 = 13x-38-8x$
$7x-12x-13x+8x = 13-5-38$
$15x-25x = 13-43$
$-10x = -30$

34. As the results are both minus, transfer each to the other side, changing their signs by the above rule.

Then $30 = 10x$, and $3 = x$.

But plainly, if $3 = x$, $x = 3$, and if $30 = 10x$, $10x = 30$. Therefore, when in any solution both sides come out minus, they may at once be made plus.

EXAMPLES FOR PRACTICE.—X.

(1.) $7x+5 = 19$.
(2.) $3x-5 = 19$.
(3.) $13x+6 = 5x+22$.
(4.) $10x-13 = 12+5x$.
(5.) $7x+3-6x = 21-5x$.
(6.) $13x-37-8x-17+4x = 0$.

35. When any of the quantities are enclosed in brackets, it is necessary to remove them before collecting, care being taken to change the signs of those terms that are within brackets preceded by minus.

Thus, $-(4x-7)$ becomes $-4x+7$.

Illustrative Example.

$$8x-4-(3+5x)=(2x-3)-(3x-4)$$
$$8x-4-3-5x=2x-3-3x+4$$
$$3x-7=1-x$$
$$4x=8, \qquad \therefore\ x=2.$$

(7.) $7x-(3x+5)=2x+1.$

(8.) $(5x+2)-8-(2-5x)=-3.$

(9.) $3-(13x+4)=x-(13-4x).$

(10.) $(8x-7)-(-1-x)-(4x+9)=0.$

(11.) $\{7x-(15-4x)\}-\{20+(2x-8)\}=9.$

(12.)

$$7x-(10x-15)=17-[4x-\{15-(3x+13)+5x\}+11].$$

Problems involving Subtraction:—

Illustrative Examples.

(1.) Three pieces of cloth measure in all 154 yards; the first is 7 yards longer than the second, and the second 9 yards longer than the third. Find the length of each.

Let x = the number of yards in first piece.

Then $x-7$ = the number of yards in second piece.

And $x-7-9$ = the number of yards in third piece.

Adding, we have $3x-23$ = number of yards altogether.

But 154 is the number of yards altogether.

$$\therefore\ 3x-23=154.$$

Transposing, $3x=154+23=177.$

$$\therefore\ x=59 = \text{number of yards in first piece.}$$
$$x-7=52 = \text{number of yards in second piece.}$$
$$x-16=43 = \text{number of yards in third piece.}$$

Proof, $59+52+43=154.$

The solution may also be obtained by putting x = number of yards in third piece; then $x+9$ will = those in second, and $x+16$ = those in third. This gives $3x+25=154$, from which $x=43$ = number of yards in third.

Similarly, x may be made = number of yards in second piece, and then $3x - 2 = 154$, from which $x = 52$ = number of yards in second.

The following question requires the use of brackets in its solution :—

(2.) A, who has three times as much money as B, gives B £10; and now, A's money falls as far short of £80 as B's does of £60. What had each at first?

Let x = number of pounds B had, then $3x$ = number A had. $\therefore x + 10$ and $3x - 10$ will equal their respective sums after A has given B £10. But the former of these is as much less than £60 as the latter is less than £80.

$$\text{Or, } 60 - (x + 10) = 80 - (3x - 10).$$

$$\text{Clearing off brackets, } 60 - x - 10 = 80 - 3x + 10$$

$$\text{Transposing, } 3x - x = 80 + 10 - 60 + 10$$

$$\text{Collecting, } 2x = 40.$$

$$\therefore x = £20 = \text{B's, and } 3x = £60 = \text{A's.}$$

EXAMPLES FOR PRACTICE.—XI.

(1.) In a parliamentary division 527 members voted, and the majority was 59. What were the numbers on each side?

(2.) A rod 29 feet long is divided into two parts so that one part is 7 feet longer than the other. Find the parts.

(3.) A congregation consisted of 950 persons; there were *three times* as many men as children, and *forty more* women than men. Required the number of women.

(4.) Divide a cord 5 feet 5 inches long into two parts such that the one shall exceed the other by 1 foot 3 inches.

(5.) A carriage with horse and harness cost together £114, 15s.; the carriage cost £34, 10s. more than the horse, and the horse £17, 5s. more than the harness. Find the price of each.

(6.) What number is that to which, if 9 be added, the sum will be 13 less than thrice the number itself?

(7.) In dividing a quantity of nuts among a number of boys, I found that if I gave 18 to each I should have 23 too few, but if I gave 16 to each I should have 19 over. How many boys were there, and how many nuts?

(8.) A is thrice as old as B, and ten years ago their united ages amounted to what A's alone does now. Find their ages.

(9.) A farmer bought a number of sheep at 25s. a head, but found himself £5, 15s. short of the amount required to pay for them; had he given only 23s. for each, he would have had £5, 15s. over. How many did he buy, and what money had he?

(10.) A traveller sets out on a journey, intending to finish it in six days, and to travel twenty miles a day, but finds from the state of the roads that he must travel each day three miles less than on the preceding one. It is required how far he must go the first day so that he may complete his journey within the given time.

(11.) A vessel, whose rate of sailing is 7 miles an hour, leaves port at midnight, and at 5 A.M. a steamer, whose rate is 10 miles, is sent to overtake it. At what time will the latter get sight of the former, if it is visible when they are 8 miles apart?

(12.) Two purses contained equal amounts. In one there were 15 crowns, 12 florins, and 17 of a certain foreign coin; in the other, 7 guineas, 6 half-crowns, and 3 of the same foreign coins. Find the value of one of the foreign pieces.

The following questions may require the aid of brackets in their solution:—

(13.) Adam has three marbles more than Peter, and Peter four more than James; now Adam and James have together as many less than 30 as Peter has less than 19. How many has each?

(14.) A mother is seven times as old as her child, and the father nine times, while the united ages of the mother and child fall short of the father's age by four years. Required the age of each.

(15.) Two numbers differ by 24, and the one is as much above 87 as the other is below it. What are they?

(16.) How long is it after midnight, when just now it is as much past 3 A.M. as five hours ago it wanted of 11 P.M.?

(17.) 50 riflemen went to the targets; in the first round the outers were twice as many as the centres, and the centres exceeded the bull's-eyes by *three*, while the misses were observed to equal the difference between the outers and the bull's-eyes. How many hits were made?

Put x = *number of centres.*

(18.) Three sisters, the eldest of whom was as much older than the second as the second was older than the third, died one after the other, beginning at the youngest, at intervals of 12 and 30 years respectively. The eldest at her death was four times, and the second at hers was twice, the age at which the youngest died. What were their ages on the death of the youngest?

CHAPTER III.

MULTIPLICATION.

36. Multiplication is indicated by the sign $\times$ (read *into*, or *multiplied by*) being placed between the quantities to be multiplied together.

Thus 3×4 means 3 multiplied by 4, and $a \times b$ means a multiplied by b.

But while 3 can be actually multiplied by 4, producing 12, a cannot be actually multiplied by b, and the result is simply indicated by placing the letters close to one another; or, $a \times b = ab$, and $a \times b \times c = abc$. A point placed between the quantities also indicates multiplication; so that $a \times b \times c$, $a \,.\, b \,.\, c$, and abc all imply the same thing—namely, the *product* of a, b, and c.

The quantities to be multiplied together are called factors, and the result is called the product.

37. When more than two quantities are multiplied together, the answer is called the *continued product:* thus, $abxy$ is the continued product of a, b, x, and y; and $(a-x)(b-y)(c-z)$ is the continued product of $a-x$, $b-y$, and $c-z$.

38. It is important to observe that factors may be multi-

plied together in any order without affecting the value of the answer. Thus: $3 \times 4 \times 5 = 3 \times 5 \times 4 = 4 \times 3 \times 5 = 4 \times 5 \times 3 = 5 \times 3 \times 4 = 5 \times 4 \times 3 = 60$; and $a \times b \times c = abc = acb = bac = bca = cab = cba$.

39. Algebraic quantities being either positive or negative, it is here necessary to inquire as to the effect on the character of the result produced by the multiplication of quantities of the same or of different signs together. For while, from the analogies of arithmetic, we may safely conclude that the multiplication of a positive quantity by a positive quantity produces a positive result, we have no guide as to the sign of the product when a negative is multiplied by a negative. Attention is therefore directed to the following *articles*.

40. $(+a) \times (+b)$ signifies that the positive quantity a is to be taken b times. If a or $+a$ be taken once, the answer is plainly $+a$; if taken twice, it is $+2a$ (Art. 3); and if taken b times, it is $+ba$ or $+ab$.

$$\therefore (+a) \times (+b) = +ab.$$

41. $(-a) \times (+b)$ signifies that the negative quantity $-a$ is to be taken b times. If $-a$ be taken once, the answer is simply $-a$; if taken twice, $-2a$ (Art. 3); and if taken b times, $-ba$ or $-ab$.

$$\therefore (-a) \times (+b) = -ab.$$

42. $(+a) \times (-b)$ implies that the positive quantity $+a$ is to be *subtracted* b times. If a be subtracted once, the answer is $-a$ (Art. 23); if twice, $-2a$; and if b times, $-ba$ or $-ab$.

$$\therefore (+a) \times (-b) = -ab.$$

43. $(-a)\times(-b)$ means that the negative quantity $-a$ is to be *subtracted* b times. If $-a$ be subtracted once, the answer is $+a$ (Art. 23); if twice, $+2a$; and if b times, $+ba$ or $+ab$.

$$\therefore\ (-a)\times(-b)=+ab.$$

44. Collecting these results, we find that a plus quantity multiplied by a plus quantity, or a minus by a minus, gives a plus product; while a minus by a plus, or a plus by a minus, gives a minus product.

More shortly: *In multiplication*, LIKE SIGNS GIVE PLUS, UNLIKE SIGNS GIVE MINUS.

45. Multiplication divides naturally into three cases:—

I. Multiplication of a simple quantity by a simple quantity, such as $3a^2b$ by $2b^2y$.

II. Multiplication of a compound quantity by a simple one, as $a^2-2ab+b^2$ by ab.

III. Multiplication of a compound quantity by a compound one, as a^3-b^3 by $a-b$.

CASE I.

46. Multiplication of a Simple Quantity by a Simple Quantity.

Multiply $5xy$ by $3ab$.

As the factors may be taken in any order (Art. 38),—

$$5xy\times 3ab=5\times 3\times ab\times xy=15abxy.$$

Here the numerical coefficients are actually multiplied together for the numerical coefficient of the answer, and the letters are placed close together in alphabetical order.

So, $7m\times 6anx=42amnx$.

47. Multiply a by a.

$a\times a=aa$; but for convenience aa is written a^2, so aaa

is written a^3, $aaaa$ is written a^4, and so on for any number of a's.

a^2, a^3, a^4, etc., are called powers of a, and are read a second power, a third power, a fourth power, etc., or sometimes a to the second, a to the third, etc.

48. The number that indicates the power to which a letter or quantity is to be raised is called the exponent or index (plural, indices), and is always written to the right of the upper part of the letter.

When a letter or quantity has no index, 1 is understood: thus, $a = a^1$.

49. Multiply a^3 by a^2.

$$a^3 = aaa, \text{ and } a^2 = aa.$$

$$\therefore\ a^3 \times a^2 = aaa \times aa = aaaaa = a^5 = a^{3+2}.$$

Here the number of a's in the product is equal to the number in the multiplier added to the number in the multiplicand.

Similarly—

$$b^2 \times b^5 \times b = bb \times bbbbb \times b = bbbbbbbb = b^8 = b^{2+5+1}.$$

From which we observe that the product of any number of powers of the same letter is obtained by adding the exponents of the powers together. We have therefore the following:—

50. RULE FOR THE MULTIPLICATION OF SIMPLE QUANTITIES.—Write down the proper sign (Art. 44). Multiply the coefficients together for a new coefficient, and place the letters in order after it, the exponents of powers of the same letter being added together.

Illustrative Examples.

(1.) $3a^2bc^3 \times 6ab^3x^2 = 3 \times 6a^{2+1}b^{1+3}c^3x^2$

$$= 18a^3b^4c^3x^2.$$

(2.) $-4a^3bx \times 2bx^2y = -4 \times 2a^{3}b^{1+1}x^{1+2}y$
$= -8a^3b^2x^3y.$

(3.) $5a^2mxy \times -3anxz = -15a^3mnx^2yz.$

(4.) $-6a^3c \times -4c^2y^3 = 24a^3c^3y^3.$

Case II.

51. Multiplication of a Compound Quantity by a Simple one.

Rule.—Multiply each term of the compound quantity separately by the simple one, as in Case I., and write the several products with their proper signs.

Illustrative Examples.

(1.) Multiply $2a^2 - 3ab + 4b^2$ by $3a^2b$.

$$\begin{array}{l} 2a^2 - 3ab + 4b^2 \\ 3a^2b \\ \hline 6a^4b - 9a^3b^2 + 12a^2b^3. \end{array}$$

(2.) Multiply $x^2y^2 - 4xy + 4$ by $-2ax^2y$.

$$\begin{array}{l} x^2y^2 - 4xy + 4 \\ -2ax^2y \\ \hline -2ax^4y^3 + 8ax^3y^2 - 8ax^2y. \end{array}$$

52. The multiplication of a compound quantity by a simple one is frequently indicated thus—

$$4a^2b(5a^2 - 10ab + 2b^2).$$

Case III.

53. Multiplication of a Compound Quantity by a Compound one.

Rule.—Multiply each term of the multiplicand by each term of the multiplier, and collect the results as in addition.

Illustrative Examples.

(1.) Multiply $a^3 - 3a^2b - 3ab^2 + b^3$ by $a^2 + 3ab - b^2$.

$$\begin{array}{l} a^3 - 3a^2b - 3ab^2 + b^3 \\ a^2 + 3ab \ - b^2 \\ \hline a^5 - 3a^4b - \ 3a^3b^2 + \ a^2b^3 \\ \quad 3a^4b - \ 9a^3b^2 - 9a^2b^3 + 3ab^4 \\ \qquad - \ \ a^3b^2 + 3a^2b^3 + 3ab^4 - b^5 \\ \hline a^5 \qquad - 13a^3b^2 - 5a^2b^3 + 6ab^4 - b^5. \end{array}$$

(2.) Multiply $ax^2 + 2bx + c$ by $dx - e$.

$$\begin{array}{l} ax^2 + 2bx + c \\ dx - e \\ \hline adx^3 + 2bdx^2 + cdx \\ \quad - aex^2 - 2bex - ce \\ \hline adx^3 + (2bd - ae)x^2 + (cd - 2be)x - ce. \end{array}$$

In this question, the coefficients of x and its powers being literal, the coefficients of *like powers* are collected as in Art. 12. The answer may also, by Art. 31, be written in the form—

$$adx^3 - (ae - 2bd)x^2 - (2be - cd)x - ce.$$

54. The multiplication of compound quantities is indicated by writing them close together in brackets, thus—

$$(a^3 - 4a^2b + 2ab^2 - b^3)(a^2 - 4ab + 4b^2).$$
$$(a^2 + ab + b^2)(a - b)(a^2 - ab + b^2)(a + b).$$
$$\{a^2 - (b - c)^2\}\{a^2 - (b + c)^2\}.$$

EXAMPLES FOR PRACTICE.—XII.

(1.) Multiply $4a^2x^3y$ by $6b^2cz$.
(2.) Multiply $5a^3b^2m$ by $7ab^2m^3x^2$.
(3.) Multiply $-9c^4x^2y^3$ by $4xyz^3$.
(4.) Multiply $11m^2n^3y^2$ by $-5m^3x^2y$.
(5.) Multiply $-12p^3qr^3s^5$ by $-2pq^2r^2s^3$.

(6.) Find the continued product of $-13a^2bx^3$, $-7ac^3y^3$, and $3b^2xyz^4$.

(7.) Multiply $3a^2b+5ab^2$ by $4ab$.

(8.) Multiply $4ax-by$ by $6a^2y^2$.

(9.) Multiply $-a^2+4ab-4b^2$ by $-3abxy$.

(10.) Multiply $3x+2y$ by $2x+3y$.

(11.) Multiply $a^2+2ab+b^2$ by $a+b$.

(12.) Multiply $3a^3-2ax-5x^3$ by $3a-x$.

(13.) Multiply $4a^2+2ax+x^2$ by $4a^2-2ax+x^2$.

(14.) Multiply $3x^2y^2-2xy+1$ by $3x^2y^2+2xy-1$.

(15.) Multiply $a^3-6a^2b+12ab^2-8b^3$ by $a-2b$.

(16.) Multiply $7x-y$ by $7y-x$.

(17.) Multiply $a+b+c$ by $a-b-c$.

(18.) Multiply—

$$4x^4+3x^3+2x^2+x+1 \text{ by } 4x^4-3x^3+2x^2-x+1.$$

(19.) Multiply $x^2+y^2+z^2-xy-xz-yz$ by $x+y+z$.

(20.) Multiply $(2a-1)^2$ by $(2-a)^2$.

(21.) Find the continued product of $x-3$, $x-2$, and $x+1$.

(22.) Also of $(am-1)^2$, $(am+1)^2$, and a^2m^2+1.

(23.) Find—

$$(a^2-2x^2)(a^2-2ax+2x^2)(a^2+2x^2)(a^2+2ax+2x^2).$$

(24.) Simplify—

$$(a+b+c)(a+b+d)+(a+c+d)(b+c+d)-(a+b+c+d)^2.$$

(25.) Multiply $x^2+2ax+a^2$ by $x+b$.

(26.) Multiply $x^2-3ax+2b$ by $x^2-2bx+3a$.

(27.) Multiply $ax+b$, $ax-c$, and $ax+d$ together.

(28.) Multiply $x-(a-b)$ by $x-(b-c)$.

(29.) Multiply $x^2-(1+m)x+b^2$ by $x^2+(1-m)x+b^2$.

(30.) Multiply $x^3-(a-b)x^2+(a-b)^2x-(a-b)^3$

by $x^3+(a+b)x^2+(a+b)^2x+(a+b)^3$.

55. Multiplication by Detached Coefficients. — When the quantities to be multiplied together contain not more than two different letters, and when the powers of these increase or decrease uniformly in the terms, considerable

trouble may be saved by using the coefficients without the letters.

Multiply $a^3 - 3a^2b + 3ab^2 - b^3$ by $3a^2 - 2ab + b^2$.

$$\begin{array}{rrrrr}
a^3 - & 3a^2b + & 3ab^2 & - b^3 & \\
3a^2 - & 2ab & + b^2 & & \\
\hline
3a^5 - & 9a^4b + & 9a^3b^2 - & 3a^2b^3 & \\
- & 2a^4b + & 6a^3b^2 - & 6a^2b^3 + 2ab^4 & \\
 & & a^3b^2 - & 3a^2b^3 + 3ab^4 - b^5 & \\
\hline
3a^5 - & 11a^4b + & 16a^3b^2 - & 12a^2b^3 + 5ab^4 - b^5. &
\end{array}$$

Here we observe that the same powers of the same letters are arranged under one another, and that the powers of a regularly decrease, while those of b regularly increase. We may, therefore, in the work leave out the letters, and only insert them in the last line, thus—

$$\begin{array}{rrrrr}
1 - & 3 + & 3 - & 1 & \\
3 - & 2 + & 1 & & \\
\hline
3 - & 9 + & 9 - & 3 & \\
- & 2 + & 6 - & 6 + 2 & \\
 & & 1 - & 3 + 3 - 1 & \\
\hline
3 - & 11 + & 16 - & 12 + 5 - 1. &
\end{array}$$

As a appears in the first term of the multiplicand raised to the third power, and in the first term of the multiplier raised to the second power, it must appear in the first term of the product in the fifth power, and in every succeeding term one power less. b does not enter into the first term, but does into each of the others, increasing uniformly by one degree in each. Bearing this in mind, the complete answer may now be written down—

$$3a^5 - 11a^4b + 16a^3b^2 - 12a^2b^3 + 5ab^4 - b^5.$$

56. Should any of the terms in either multiplier or multiplicand of what would otherwise be a series, be omitted, their places may be filled by 0, which, indeed, forms their coefficient.

Multiply $x^5 - 3x^2 - 2x + 1$ by $x^2 - 1$.

$$\begin{array}{r} 1+0+0-3-2+1 \\ 1+0-1 \\ \hline 1+0+0-3-2+1 \\ \quad -1+0+0+3+2-1 \\ \hline 1+0-1-3-2+4+2-1 \end{array}$$

$$x^7 - x^5 - 3x^4 - 2x^3 + 4x^2 + 2x - 1.$$

The whole of the questions in the preceding Examples for Practice, with the exception of Nos. 17, 19, and 24, may be wrought by this method.

The student is recommended to do them.

57. Theorems in Multiplication. — There are certain results in multiplication which it is of great importance to keep in mind. They form a sort of multiplication table by the help of which many algebraical calculations may be much simplified and shortened.

I. Multiply $a + b$ by itself.

$$\begin{array}{r} a + b \\ a + b \\ \hline a^2 + ab \\ \quad ab + b^2 \\ \hline a^2 + 2ab + b^2 \end{array}$$

$$\therefore (a+b)^2 = a^2 + 2ab + b^2.$$

That is, the square of the sum of two quantities is equal to the sum of the squares of the quantities increased by twice their product.

II. Multiply $a - b$ by itself.

$$\begin{array}{r} a - b \\ a - b \\ \hline a^2 - ab \\ \quad - ab + b^2 \\ \hline a^2 - 2ab + b^2 \end{array}$$

$$\therefore (a-b)^2 = a^2 - 2ab + b^2.$$

Or, the square of the difference of two quantities is equal to the sum of their squares diminished by twice their product.

III. Multiply $a+b$ by $a-b$.

$$\begin{array}{r} a+b \\ a-b \\ \hline a^2+ab \\ \quad -ab-b^2 \\ \hline a^2 \qquad -b^2 \end{array}$$

$$\therefore (a+b)(a-b)=a^2-b^2.$$

That is, the product of the sum and difference of two quantities is equal to the difference of the squares of the quantities.

IV. Multiply a^2-ab+b^2 by $a+b$.

$$\begin{array}{r} a^2-ab+b^2 \\ a+b \\ \hline a^3-a^2b+ab^2 \\ \quad a^2b-ab^2+b^3 \\ \hline a^3 \qquad\qquad +b^3 \end{array}$$

$$\therefore (a^2-ab+b^2)(a+b)=a^3+b^3.$$

That is, if the sum of the squares of two quantities diminished by their product be multiplied by their sum, the result is the sum of their cubes.

V. Multiply a^2+ab+b^2 by $a-b$.

$$\begin{array}{r} a^2+ab+b^2 \\ a-b \\ \hline a^3+a^2b+ab^2 \\ \quad -a^2b-ab^2-b^3 \\ \hline a^3 \qquad\qquad -b^3 \end{array}$$

That is, if the sum of the squares of two quantities, increased by their product, be multiplied by their difference, the result is the difference of their cubes.

The following, although not so important, will often be found useful :—

VI. $(a^2+ab+b^2)(a^2-ab+b^2)=a^4+a^2b^2+b^4$.

VII. $(a+b)^3=a^3+3a^2b+3ab^2+b^3$.

VIII. $(a-b)^3=a^3-3a^2b+3ab^2-b^3$.

IX. $(a+b+c)^2=a^2+b^2+c^2+2(ab+ac+bc)$.

X. $(x\pm a)(x\pm b)=x^2\pm(a+b)x+ab$.

XI. $(x\pm a)(x\mp b)=x^2\pm(a-b)x-ab$.

XII. $(x+a)(x+b)(x+c)=x^3+(a+b+c)x^2+(ab+ac+bc)x+abc$.

In X. and XI. either the two upper or the two under signs must be taken together, not an upper and an under at the same time.

Application of Theorems.

Illustrative Examples.

(1.) $(3x^2+2y)^2=(3x^2)^2+2(3x^2)(2y)+(2y)^2$, By Theorem I.
$=9x^4+12x^2y+4y^2$.

(2.) $(a+3b-4c)^2=\{(a+3b)-4c\}^2$
$=(a+3b)^2-2(a+3b)(4c)+(4c)^2$, II.
$=a^2+6ab+9b^2-8ac-24bc+16c^2$. I.

(3.) $(x^2+2x+1)(x^2-2x+1)=\overline{(x^2+1}+2x)(\overline{x^2+1}-2x)$,
$=(x^2+1)^2-(2x)^2$, III.
$=x^4+2x^2+1-4x^2$, I.
$=x^4-2x^2+1$.

(4.) $(x^2-2x+4)(x+2)=x^3+8$. IV.

(5.) $(4x^2+6x+9)(2x-3)=8x^3-27$. V.

(6.) $(4x^2+6x+9)(4x^2-6x+9)(16x^4-36x^2+81)$,
$=(16x^4+36x^2+81)(16x^4-36x^2+81)$, VI.
$=256x^8+1296x^4+6561$. VI.

(7.) $(x^2+2x+3)^3=\{x^2+(2x+3)\}^3$,
$=(x^2)^3+3(x^2)^2(2x+3)+3(x^2)(2x+3)^2+(2x+3)^3$, VII.
$=x^6+6x^5+9x^4+12x^4+36x^3+27x^2+8x^3+36x^2+$
$54x+27$, VII. and I.
$=x^6+6x^5+21x^4+44x^3+63x^2+54x+27$.

(8.) $(2a-b-c)^3=\{(2a-b)-c\}^3$,
$=(2a-b)^3-3(2a-b)^2c+3(2a-b)c^2-c^3$, Theorem VIII.
$=8a^3-12a^2b+6ab^2-b^3-12a^2c+12abc-3b^2c+6ac^2-3bc^2-c^3$, VIII. and II.

(9.) $(5a+3b+c)^2=25a^2+9b^2+c^2+2(15ab+5ac+3bc)$. IX.

(10.) $(x+7)(x+5)=x^2+12x+35$. X.

(11.) $(x-15)(x+4)=x^2-11x-60$. XI.

(12.) $(x+2)(x-2)(x-6)=x^3-6x^2-4x+24$. XII.

EXAMPLES FOR PRACTICE.—XIII.

Expand the following by the above theorems :—

(1.) $(7a+2bc)^2$.
(2.) $(5x^2-y^2)^2$.
(3.) $(x+2y+3z)^2$.
(4.) $(x^2-2x+1)^2$.
(5.) $(x^2+7)(x^2-7)$.
(6.) $(x^2+2x+4)(x^2-2x+4)$.
(7.) $(x^2+xy-y^2)(x^2-xy+y^2)$.
(8.) $(x-1)(x+1)(x^2+1)(x^4+1)$.
(9.) $(mx+ny)^2(mx-ny)^2$.
(10.) $(a+b+c+d)(a+b-c-d)$.
(11.) $(1+2x-2x^2-x^3)(1-2x+2x^2-x^3)$.
(12.) $(1-2x+4x^2)(1+2x)$.
(13.) $(a^2b^2+abc+c^2)(ab-c)$.
(14.) $(a^2-4a-4)(a-2)^2$.
(15.) $(x^2+6x-9)(x-3)^2-(x^2+9)(x+3)(x-3)$.
(16.) $(4x^2-6xy+9y^2)(2x+3y)-(4x^2+10xz+25z^2)(2x-5z)-(9y^2-15yz+25z^2)(3y+5z)$.
(17.) $(a^2x^2+abxy+b^2y^2)(a^2x^2-abxy+b^2y^2)$.
(18.) $(x^2+3x+9)(x^2-3x+9)(x^4-9x^2+81)$.
(19.) $(1+2xy)^3-(1-3xy)^3-5xy(1-xy)(3-7xy)$.
(20.) $5(x+4)(x+5)-7(x-2)(x-3)+2(x+5)(x-3)$.
(21.) $(x+7)(x+5)(x+3)-(x-3)(x-5)(x-7)$.
(22.) $(x+4)(x+3)(x-2)+(x-6)(x+2)(x-2)$.

(23.) $(x+2a)(x-b)-(x-a)(x+2b)$.

(24.) $\{x^2+(a-b)x+(a-b)^2\}(x-a+b)+$
$\{x^2-(a+b)x+(a+b)^2\}(x+a+b)$.

58. Substitutions.—Find the value of the following expressions when $a=13$, $b=7$, $c=3$, $d=0$, $x=4$, $y=5$, and $z=9$.

Illustrative Example.

$$(a+b)(x^2-y)-(2y-z)^2+ab(x-y)+cd(y-z),$$
$$=(13+7)(16-5)-(10-9)^2+13\times7\times(4-5)+3\times0\times(5-9),$$
$$=20\times11-1^2+13\times7\times-(1)+0,$$
$$=220-1-91=220-92=128.$$

EXAMPLES FOR PRACTICE.—XIV.

The values of the letters being the same as above.

(1.) $abx-bcy+cdz$.

(2.) $3x^2-5xy+7y^2$.

(3.) $5(a+b)x+4(b+c)y-11(c+d)z$.

(4.) $x^4-4x^3y+6x^2y^2-4xy^3+y^4$.

(5.) $a\{b+c(x-y)\}-2[a-3\{b-4(c-x)-2y\}+z]$.

(6.) $\{a(x+y)-b(x-y)\}\{c(y+z)-d(y-z)\}-(ax-by+cz)^2$.

59. Equations involving Multiplication.

Illustrative Examples.

(1.) Given $4(x+2)-3(2-x)=5(x+6)$ to find x

Solution :—

$$4x+8-6+3x=5x+30. \qquad (a)$$
$$7x-5x=30-2.$$
$$2x=28, \therefore x=14.$$

The student must carefully remember the change of sign necessary when brackets preceded by minus are removed, as in line marked (a).

(2.) $(x-1)(x-4)=(x-2)(x-5)$. Find x.

$$x^2-5x+4=x^2-7x+10.$$
$$2x=6,\ \therefore\ x=3.$$

(3.) $(x-1)(x-2)(x-3)+x(x+2)(x+4)=2(x^3+16)$.

$$x^3-6x^2+11x-6+x^3+6x^2+8x=2x^3+32.$$
$$2x^3+19x-6=2x^3+32.$$
$$19x=38,\ \therefore\ x=2.$$

(4.) $\{(x-3)^2+a(x-3)+a^2\}(x-3-a)=\{(x-2)^2-b(x-2)+b^2\}(x-2+b)-(3x-2)(x-4)$.

$$(x-3)^3-a^3=(x-2)^3+b^3-(3x^2-14x+8).$$
$$x^3-9x^2+27x-27-a^3=x^3-6x^2+12x-8+b^3-3x^2+14x-8.$$
$$=x^3-9x^2+26x\quad b^3-16.$$
$$\therefore\ x=a^3+b^3+11.$$

60. If, after expansion, the powers of x higher than the first do not all disappear, the equation is not a simple one, and cannot be solved until the pupil is more advanced. For instance, $(3x+1)(x-2)=(x+1)(x+3)$, when expanded and transposed, gives $2x^2-9x=5$, an equation containing the second power of x.

EXAMPLES FOR PRACTICE.—XV.

(1.) $2(2x-9)=x-6$.
(2.) $3(5-3x)-5(3-5x)=8(x+2)$.
(3.) $x(4x+3)-(2x-1)^2=2(2x+7)$.
(4.) $5x-(x-2)x+(x-1)(x+1)=20$.
(5.) $8\{x-3(5-2x)\}-7(3x+5)=17x-29$.
(6.) $6x^2-5\{(x-4)(x-3)-5(x-2)\}=(x+10)^2$.
(7.) $(x-1)(x+1)(x+3)=(x-4)(x-2)(x+9)$.
(8.) $(x^2+2x+2)(x^2-2x+2)-(x^2+2)(x^2-2)=x$.
(9.) $(x^2+x+1)(x-1)-(x+1)^2(x-3)=x^2+7$.
(10.) $(2x-3)^3-(2x+1)^3=12x-1-3(4x-1)^2$.

(11.) $(3+x)(2-x)(1-x)-(3-x)(2+x)(1-x)=(3-x)(2-x)(1+x)+(3-x)(2-x)(1-x)$.

(12.) $2\{(x+1)^2+(x-2)^2\}\{(x+1)^2-(x-2)^2\}=3(2x-1)^3-9(1-4x)$.

61. Problems involving Multiplication.

Illustrative Examples.

(1.) Divide 40 into two parts, such that three times one part may equal five times the other.

Let x = the one part,
Then $40-x$ = the other.

And by the terms of the question—

$$3x=5(40-x).$$

Or, multiplying the quantity within the brackets by 5—

$$3x=200-5x.$$

Transposing, $3x+5x=200.$

Collecting, $8x=200.$

$\therefore\ x=25$, and $40-x=15$.

(2.) The product of two numbers which differ by 9 is equal to the square of the number which is 6 less than the greater. Find the numbers.

Let x = the greater, then $x-9$ = the less,
And $x-6$ = number 6 less than greater.
By question, $x(x-9)=(x-6)^2$

$$x^2-9x=x^2-12x+36.$$

Here x^2, appearing on each side with the same sign, may be struck out.

Transposing, $12x-9x=36.$

$$3x=36.$$

$\therefore\ x=12$, and $x-9=3$.

By making x = the less number, and $x+9$ the greater, we would have the equation, $(x+9)x=(x+3)^2$, from which we would find $x=3$, and $x+9=12$.

(3.) A traveller found after walking 3 hours that he had gone 2 miles more than one-third of his entire journey, and after 4 hours that he had done one mile more than half. At what rate per hour did he walk, and what was the length of his journey?

Let x = number of miles he went in an hour.

Then $3x$ and $4x$ = number travelled in 3 and 4 hours respectively.

$$\begin{aligned} 3x - 2 &= \text{one-third of journey.} \\ 3(3x-2) &= \text{whole journey.} \\ 4x - 1 &= \text{one-half of journey.} \\ 2(4x-1) &= \text{whole journey.} \\ \therefore\ 3(3x-2) &= 2(4x-1) \\ 9x - 6 &= 8x - 2. \end{aligned}$$

Transposing and collecting, $x = 4$.

$2(4x-1) = 2(16-1) = 30 =$ number of miles in journey.

(4.) A rectangle is 7 feet more in length than in breadth; if both length and breadth be increased by 3 feet, the area will be increased by 96 square feet. Find the area.

$$\begin{aligned} \text{Let}\quad x &= \text{number of feet in breadth.} \\ x+7 &= \text{number of feet in length.} \\ x(x+7) &= \text{area.} \end{aligned}$$

If length and breadth be each increased by 3, they will become $x+3$, and $x+10$, and the area will be

$$(x+3)(x+10).$$

But by the question, $(x+3)(x+10) = x(x+7) + 96.$

$$x^2 + 13x + 30 = x^2 + 7x + 96.$$

Transposing and collecting, $6x = 66$, $\therefore\ x = 11$, and

$$x(x+7) = 11 \times 18 = 198 = \text{area in square feet.}$$

(5.) A cistern is supplied by four pipes, the first conveying 10 gallons a minute more than the second, the second 10 more than the third, and the third 10 more than the fourth. The second running 30 minutes, the

third running 20, and the fourth 10, will together fill the cistern, while the first can do it alone in 40 minutes. How much does the cistern hold?

Let x = number of gallons conveyed per minute by first.
$x - 10$ = number of gallons conveyed per minute by second.
$x - 20$ = number of gallons conveyed per minute by third.
$x - 30$ = number of gallons conveyed per minute by fourth.

Then $40x$, $30(x-10)$, $20(x-20)$, and $10(x-30)$ represent the total quantities conveyed by the respective pipes; and by the question—

$$30(x-10)+20(x-20)+10(x-30)=40x.$$
$$30x-300+20x-400+10x-300=40x.$$

Transposing and collecting, $20x=1000$, and $x=50$.

$40x=40\times 50=2000$ = number of gallons in cistern.

(6.) On looking over a farmer's stock, it was found that the number of sheep exceeded the number of shillings an ox was worth by 460; that the number of oxen exceeded the number of shillings a sheep was worth by 25; and that the total value of the oxen exceeded the total value of the sheep by £100, although there were twenty times as many sheep as oxen. Find the number of oxen, and the value of the farmer's stock.

Let x = number of oxen.
$20x$ = number of sheep.
$20x-460$ = number of shillings in price of ox.
$x-25$ = number of shillings in price of sheep.
$x(20x-460)$ = total value of oxen in shillings.
$20x(x-25)$ = total value of sheep in shillings.

By question, $x(20x-460)-20x(x-25)=2000$s.

$$20x^2-460x-20x^2+500x=2000.$$

$\therefore\ 40x=2000,\ x=50$ = number of oxen.

$x(20x-460)+20x(x-25)=50\times 540+20\times 50\times 25$
$=52000$s. = £2600 = value of stock.

EXAMPLES FOR PRACTICE.—XVI.

(1.) If 11 be added to a certain number, thrice the sum will equal 57 : find the number.

(2.) A has seven times as many marbles as B, but if each receive an addition of 10, A's will only number thrice B's. What number has each?

(3.) The product of two consecutive numbers is greater than the square of the less by 7. Find the numbers.

(4.) Two pieces of cloth, together measuring 69 yards, were bought for £17, 10s. 9d. ; one of the pieces cost 6s. 3d. per yard, and the other 4s. 6d. How many yards were in each?

(5.) A rectangle is twice as long as it is broad. Another, of equal area, is 10 feet shorter but 6 feet broader. Find the length of each, and the common area.

(6.) A cistern which contains 1280 gallons is filled in forty minutes by three pipes, one conveying 5 gallons more and another 3 gallons less than the third per minute. How many gallons does each contribute to fill the cistern?

(7.) Find two numbers such that their difference may be 6, and the difference of their squares 168.

(8.) A lady distributed £4, 1s. among 45 poor persons, giving 2s. 6d. to a man, 1s. 6d. to a woman, and 9d. to a child. There were thrice as many men as children. How many were there of each?

(9.) In the last question, if the sum given to each be increased by the same number of pence, and if for every additional penny each receives the number of men be reduced by one, the number of women by two, and the number of children be increased by three, the total sum distributed would amount to £4, 3s. 6d. How much additional should each receive?

(10.) A train passing from one terminus to another, 50

miles distant, made three stoppages by the way at stations respectively 16, 30, and 42 miles from the starting place. At the first station the number of passengers in the train was reduced by 12; at the second by 5 more; and the number who came in at the third exceeded the number who went out by 7. The fare being 2d. per mile, the whole sum paid by the various passengers amounted to £66, 13s. How many started with the train?

(11.) A regiment of volunteers being formed into a solid square, 40 men were left over; but on trying to add one to each side, it was found that there were 11 too few. How many were in the regiment?

(12.) A traveller found, after walking four hours, that he had gone one mile less than a third of his journey, and after seven hours that he had done two miles more than half. How many miles an hour did he walk?

(13.) Three pipes convey water to a cistern which can be filled by all running together for an hour, or by one after the other running for 40, 70, and 83 minutes respectively. The first discharges per minute 3 gallons more than the second, and the second 7 more than the third. How many gallons does the cistern hold?

(14.) In the last question, if the discharge of each pipe be increased by the same amount, and if, for every additional gallon discharged per minute, the first and second be allowed to run one minute more, and the third two minutes less, when they run one after the other, the cistern would overflow to the extent of 1050 gallons. What additional quantity must be discharged by each per minute?

(15.) A traveller sets out on a journey which he can complete in 9 days. Some time after, another, who can travel five miles more per day, sets out after him, and overtakes him in 6 days, 24 miles from his journey's end.

How many miles per day can each travel, and what is the length of the journey?

(16.) A square court has a walk three feet wide formed round it. If the area of the walk be 324 square feet, what is its length?

(17.) A cricket match was lost by 29 runs; but the best player on the losing side beat the best player on the other by 5, and made one-fourth of the whole number gained by his side. Had the best player on the winning side made 3 more, his runs would have equalled a fifth of those made by all the other players on his side. What was the winning score?

Put x = number of runs by best on winning side.

(18.) A farmer owns 20 times as many sheep as oxen. The number of sheep exceeds the number of shillings one of them costs by 350, and the number of oxen is one more than the number of pounds an ox costs. If there were as many cattle as there are sheep, the total value of the former would exceed the total value of the latter by £6270. How many are there of each, and what is the value of the whole?

CHAPTER IV.

DIVISION.

62. Division is indicated by the sign $\div$ (read *divided by*) being placed after the quantity to be divided, and before the one to be used as a divisor—thus: $12a^2b^3 \div 4ab$ signifies that $12a^2b^3$ is to be divided by $4ab$, and $(3a^2x - 2b^2y) \div 5abxy$ means that $3a^2x - 2b^2y$ is to be divided by $5abxy$.

Division is also very generally represented by placing the quantity to be divided above the divisor in the form of a fraction, as—

$$\frac{7a^2}{4}, \quad \frac{10a^3m^2}{2am}, \quad \frac{6a^2 - 3b^2}{3ab}, \quad \frac{4x^2 - 4xy + y^2}{2x - y},$$

in all of which the upper line is to be divided by the lower.

This subject is conveniently treated under the three cases—

I. Division of one simple quantity by another.
II. Division of a compound quantity by a simple one.
III. Division of one compound quantity by another.

Case I.

63. Division of one Simple Quantity by another.

(1.) When the divisor is merely a number, divide the coefficient of the dividend by the number.

If the coefficient will not divide by the number, express the result in the form of a fraction.

We have already frequently made use of this rule in solving equations.

Illustrative Examples.

(1) $12a^3b^2x \div 3 = 4a^3b^2x$. (2) $5ab^2x^3 \div 8 = \frac{5}{8}ab^2x^3$, or $\frac{5ab^2x^3}{8}$.

(3) $a^2b^2x^2 \div 4 = \frac{1}{4}a^2b^2x^2$. (4) $15abx \div 10 = \frac{15}{10}abx = \frac{3}{2}abx$.

64. (2.) When the divisor and the dividend consist of the same letter having different powers.

Divide a by a.

As any quantity is contained in itself once, the quotient here is evidently 1, or $\frac{a}{a} = 1$.

Similarly, $\frac{a^2}{a^2} = 1$, $\frac{a^3}{a^3} = 1$, etc.

Divide a^3 by a.

By Art. 49 we know that $a^2 \times a$ or $a^2 \cdot a$ is equal to a^3. If then a^3, that is, $a^2 \cdot a$, be divided by a, or have the factor a removed, the quotient must be a^2; that is—

$$\frac{a^3}{a} = a^2. \quad \text{So } \frac{x^5}{x^2} = \frac{x^3 \cdot x^2}{x^2} = x^3.$$

Divide a^2 by a^6.

$\frac{a^2}{a^6} = \frac{a^2}{a^4 \cdot a^2} = \frac{1}{a^4}$. In this question, as a^2 is a lower power of a than a^6, we divide out or cancel a^2 from both divisor and dividend, and leave the result in the form of a fraction.

$$\text{So also } \frac{x^3}{x^4} = \frac{x^3}{x^3 \cdot x} = \frac{1}{x}.$$

In each of these results it is to be observed that the

exponent of the quotient is equal to the difference between the exponents of the dividend and divisor.

$$\frac{a^3}{a} = a^2 = a^{3-1}, \qquad \frac{x^5}{x^2} = x^3 = x^{5-2},$$

$$\frac{a^2}{a^6} = \frac{1}{a^4} = \frac{1}{a^{6-2}}, \qquad \text{and} \qquad \frac{x^3}{x^4} = \frac{1}{x} = \frac{1}{x^{4-3}}.$$

65. From this we have the following

RULE FOR THE DIVISION OF ONE POWER OF A QUANTITY BY ANOTHER.—Find the difference of the exponents. This difference is the exponent of the quotient.

If the exponent of the divisor is greater than that of the dividend, write the answer in the form of a fraction having 1 for its numerator.

Illustrative Examples.

$$\frac{a^8}{a^5} = a^{8-3} = a^5, \qquad \text{and} \qquad \frac{a^5}{a^7} = \frac{1}{a^{7-5}} = \frac{1}{a^2}.$$

66. (3.) When the divisor and the dividend contain more than one letter.

Divide a^4b^2c by a^2bc; that is, remove the factors a^2bc from the product a^4b^2c (Art. 49).

$\frac{a^4}{a^2}$ gives a^2, $\frac{b^2}{b}$ gives b, and $\frac{c}{c}$ gives 1.

$$\therefore \frac{a^4b^2c}{a^2bc} = \frac{a^4}{a^2} \times \frac{b^2}{b} \times \frac{c}{c} = a^2 \times b \times 1 = a^2b.$$

$$\text{So also } \frac{m^2n}{m^3n^3} = \frac{1}{m} \times \frac{1}{n^2} = \frac{1}{mn^2};$$

$$\text{And } \frac{x^2y^4z^2}{x^2y^2z^4} = 1 \times y^2 \times \frac{1}{z^2} = \frac{y^2}{z^2}.$$

Divide abc by def.

As there are here no factors common to divisor and dividend, the result can only be indicated thus, $\frac{abc}{def}$.

Divide $a^2b^3c^4d^2$ by $a^2c^4e^2f^3$.

$$\frac{a^2b^3c^4d^2}{a^2c^4e^2f^3} = \frac{a^2c^4}{a^2c^4} \times \frac{b^3d^2}{e^2f^3} = 1 \times \frac{b^3d^2}{e^2f^3} = \frac{b^3d^2}{e^2f^3}.$$

67. From these different examples we deduce the

GENERAL RULE FOR THE DIVISION OF ONE SIMPLE QUANTITY BY ANOTHER.—Place the divisor under the dividend, and strike out from each the common factors.

Illustrative Examples.

Divide (1.) $12a^5b^2x^4$ by $16a^2x^3$, and (2.) $10ax^3y^2$ by $15a^2y^2z$.

$$(1.)\ \frac{12a^5b^2x^4}{16a^2x^3} = \tfrac{3}{4}a^3b^2x.$$

$$(2.)\ \frac{10ax^3y^2}{15a^2y^2z} = \frac{2x^3}{3az}$$

68. Signs.—In division, as in multiplication, it is essential to observe the effect of the signs.

By Arts. 40, 43, $+ab$ is the product of either $+a$ by $+b$, or of $-a$ by $-b$.

If, therefore, $+a$ be one of the factors, $+b$ must be the other; and if $-a$ be one of the factors, $-b$ must be the other.

$$\text{Or, } \frac{+ab}{+a} = +b, \text{ and } \frac{+ab}{-a} = -b.$$

That is, if a positive quantity be divided by a positive, the tient is positive; if by a negative, the quotient is tive.

69. By Arts. 41, 42, $-ab$ is the product of either $+a$ by $-b$, or $-a$ by $+b$.

If $+a$ be one of the factors, $-b$ must be the other; and if $-a$ be one of the factors, $+b$ must be the other.

$$\text{Or,} \quad \frac{-ab}{+a} = -b, \text{ and } \frac{-ab}{-a} = +b.$$

That is, if a negative quantity be divided by a positive, the quotient is negative; if by a negative, the quotient is positive

70. Bringing these results together, for the sake of comparison, we have—

$$(1.)\ \frac{+ab}{+a} = +b. \qquad (2.)\ \frac{+ab}{-a} = -b.$$

$$(3.)\ \frac{-ab}{+a} = -b. \qquad (4.)\ \frac{-ab}{-a} = +b.$$

From which we observe that the rule of signs is the same in division as in multiplication, namely :—

LIKE SIGNS GIVE PLUS; UNLIKE SIGNS GIVE MINUS.

Illustrative Examples.

Divide—

(1.) $12a^2b^2$ by $4a^2b$. (2.) $8ab^2c^3$ by $-6a^3b^2c$.

(3.) $-a^4b^2x$ by $5a^2b^4y$. (4.) $-3ab^3$ by $-b^2cx^2$.

$$(1.)\ \frac{12a^2b^2}{4a^2b} = 3b. \qquad (2.)\ \frac{8ab^2c^3}{-6a^3b^2c} = -\frac{4c^2}{3a^2}.$$

$$(3.)\ \frac{-a^4b^2x}{5a^2b^4y} = -\frac{a^2x}{5b^2y}. \qquad (4.)\ \frac{-3ab^3}{-b^2cx^2} = \frac{3ab}{cx^2}.$$

CASE II.

71. Division of a Compound Quantity by a Simple one.

RULE.—Divide each term of the compound quantity by the simple one, and set down the sum of the results.

Illustrative Examples.

(1.) Divide $16a^4b^3 - 12a^3b^2 + 8a^2b$ by $4a^2b$.

$$\frac{16a^4b^3 - 12a^3b^2 + 8a^2b}{4a^2b} = 4a^2b^2 - 3ab + 2.$$

(2.) Divide $12x^3y - 7x^2y^2 - 9xy^3z$ by $-6xy^2$.

$$\frac{12x^3y - 7x^2y^2 - 9xy^3z}{-6xy^2} = -2\frac{x^2}{y} + \tfrac{7}{6}x + \tfrac{3}{2}yz.$$

(3.) Divide $abc + bcd - cde$ by $abcde$.

$$\frac{abc + bcd - cde}{abcde} = \frac{1}{de} + \frac{1}{ae} - \frac{1}{ab}.$$

Case III.

72. Division of one Compound Quantity by another.

Rule.—Arrange each of the quantities according to either the descending or ascending powers of the same letter.

Divide the first term of the dividend by the first term of the divisor; this gives the first term of the quotient.

Multiply the divisor by this term.

Place the result under the dividend, and subtract.

Treat the remainder as a new dividend, and proceed as before until there is no remainder, or until the leading term of the remainder will no longer divide by that of the divisor, in which case the remainder is to be written over the divisor, in the form of a fraction, with the proper sign prefixed.

Illustrative Examples.

(1.) Divide $a^3 - x^3 - 3a^2x + 3ax^2$ by $a - x$.

Arrange according to descending powers of a.

$$\begin{array}{l|l} a - x & a^3 - 3a^2x + 3ax^2 - x^3 \\ & a^3 - a^2x \\ \hline & -2a^2x + 3ax^2 - x^3 \\ & -2a^2x + 2ax^2 \\ \hline & \quad + ax^2 - x^3 \\ & \quad + ax^2 - x^3 \end{array} \quad \Big(a^2 - 2ax + x^2$$

Here $\frac{a^3}{a}$ gives a^2, the first term of the quotient, and $a - x$ multiplied by a^2 gives $a^3 - a^2x$, which, being written under the dividend and subtracted from it, leaves the remainder, $-2a^2x + 3ax^2 - x^3$. This being treated as a new dividend, $\frac{-2a^2x}{a}$ gives $-2ax$, the second term of the quotient. Then $a - x$, being multiplied by $-2ax$, produces $-2a^2x + 2ax^2$, which, being taken from the last dividend, leaves $ax^2 - x^3$.

This remainder is now the dividend, and $\frac{ax^2}{a}$ gives x^2, the third term of the quotient.

$a - x$ multiplied by x^2 gives $ax^2 - x^3$, which, being taken from the last dividend, leaves no remainder.

The complete quotient is $a^2 - 2ax + x^2$.

(2.) Divide $18x^5 - 12x^4 + x^2 - 1$ by $3x^2 - 2x + 1$.

$$
\begin{array}{r|l}
3x^2 - 2x + 1 \;)\; 18x^5 - 12x^4 + x^2 - 1 & (\; 6x^3 - 2x - 1 \\
\underline{18x^5 - 12x^4 + 6x^3} & \\
-6x^3 + x^2 - 1 & \\
\underline{-6x^3 + 4x^2 - 2x} & \\
-3x^2 + 2x - 1 & \\
\underline{-3x^2 + 2x - 1} & \\
\end{array}
$$

Here $\frac{18x^5}{3x^2}$ gives $6x^3$, and the product of this term and the divisor is $18x^5 - 12x^4 + 6x^3$, which, being taken from the dividend, leaves $-6x^3 + x^2 - 1$. Observe that the term $6x^3$ in the subtrahend, having no *like* term in the line above it, is merely brought down with its sign changed. Also, it is brought down before the x^2, as it is necessary to keep the order of the powers the same as at first. The rest of the work is performed in a manner similar to that in last example.

(3.) Divide $8x^4y^4 + 6x^3y^3 + 3x^2y^2 - 3xy + 6$ by $2x^2y^2 + 3xy + 2$.

$$
\begin{array}{l}
2x^2y^2+3xy+2\,\big)\,8x^4y^4+6x^3y^3+3x^2y^2-3xy+6\,\big(\,4x^2y^2-3xy+2 \\
\qquad\qquad 8x^4y^4+12x^3y^3+8x^2y^2 \\
\qquad\qquad\qquad\qquad -6x^3y^3-5x^2y^2-3xy \\
\qquad\qquad\qquad\qquad -6x^3y^3-9x^2y^2-6xy \\
\qquad\qquad\qquad\qquad\qquad\qquad +4x^2y^2+3xy+6 \\
\qquad\qquad\qquad\qquad\qquad\qquad +4x^2y^2+6xy+4 \\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad -3xy+2 \qquad \text{(A)}
\end{array}
$$

The division of this sum is carried on as in the two previous examples, until we come to the line marked (A), where it is found that the exponent of the leading term in the remainder is less than that of the first term of the divisor. The division, therefore, is no longer carried on, but the remainder, with the divisor under it in the form of a fraction, is written after the other terms of the quotient, so that the complete answer is—

$$4x^2y^2-3xy+2+\frac{-3xy+2}{2x^2y^2+3xy+2}$$

Or, $$4x^2y^2-3xy+2-\frac{3xy-2}{2x^2y^2+3xy+2}.$$ (Art. 27.)

(4.) Divide $1-2ax+4a^3x^3-6a^4x^4$ by $1-3a^2x^2$.

$$
\begin{array}{l}
1-3a^2x^2\,\big)\,1-2ax+4a^3x^3-6a^4x^4\,\big(\,1-2ax+3a^2x^2 \\
\qquad\qquad 1-3a^2x^2 \\
\qquad\qquad\qquad -2ax+3a^2x^2+4a^3x^3 \\
\qquad\qquad\qquad -2ax+6a^3x^3 \\
\qquad\qquad\qquad\qquad +3a^2x^2-2a^3x^3-6a^4x^4 \\
\qquad\qquad\qquad\qquad +3a^2x^2-9a^4x^4 \\
\qquad\qquad\qquad\qquad\qquad\qquad -2a^3x^3+3a^4x^4
\end{array}
$$

The terms are here arranged in ascending powers of ax, and it is essential that in each of the remainders the lowest power of ax should stand first. As the leading term of each successive remainder has a higher power than that of the preceding one, it is obvious that so long as

there is a remainder its leading term must always be divisible by that of the divisor, and the division becomes interminable.

We can stop, therefore, at any term we please, and, *generally*, we may do so when all the terms of the dividend have been brought down. The answer to the above sum is then—

$$1 - 2ax + 3a^2x^2 - \frac{2a^3x^3 - 3a^4x^4}{1 - 3a^2x^2} .$$

Carrying it on a few terms further, we have—

$$\begin{array}{l}
1 - 3a^2x^2\big) - 2a^3x^3 + 3a^4x^4 \big(- 2a^3x^3 + 3a^4x^4 - 6a^5x^5 + 9a^6x^6 - \text{etc.} \\
\quad\quad - 2a^3x^3 + 6a^5x^5 \\
\quad\quad\quad\quad \overline{\quad 3a^4x^4 - 6a^5x^5} \\
\quad\quad\quad\quad\quad 3a^4x^4 - 9a^6x^6 \\
\quad\quad\quad\quad\quad\quad \overline{\quad - 6a^5x^5 + \ 9a^6x^6} \\
\quad\quad\quad\quad\quad\quad\quad - 6a^5x^5 + 18a^7x^7 \\
\quad\quad\quad\quad\quad\quad\quad\quad \overline{\quad 9a^6x^6 - 18a^7x^7} \\
\quad\quad\quad\quad\quad\quad\quad\quad\quad 9a^6x^6 - 27a^8x^8 \\
\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad \overline{\quad - 18a^7x^7 + 27a^8x^8}
\end{array}$$

The complete answer is now—

$$1 - 2ax + 3a^2x^2 - 2a^3x^3 + 3a^4x^4 - 6a^5x^5 + 9a^6x^6 - \frac{18a^7x^7 - 27a^8x^8}{1 - 3a^2x^2}$$

(5.) Divide 1 by $1 - 2x + x^2$.

This is of a character similar to the last.

$$\begin{array}{l}
1 - 2x + x^2\big) 1 \qquad \big(1 + 2x + 3x^2 + 4x^3 + \dfrac{5x^4 - 4x^5}{1 - 2x + x^2} \\
\quad\quad \overline{\quad 1 - 2x + x^2} \\
\quad\quad\quad 2x - x^2 \\
\quad\quad\quad \overline{\quad 2x - 4x^2 + 2x^3} \\
\quad\quad\quad\quad 3x^2 - 2x^3 \\
\quad\quad\quad\quad \overline{\quad 3x^2 - 6x^3 + 3x^4} \\
\quad\quad\quad\quad\quad 4x^3 - 3x^4 \\
\quad\quad\quad\quad\quad \overline{\quad 4x^3 - 8x^4 + 4x^5} \\
\quad\quad\quad\quad\quad\quad 5x^4 - 4x^5
\end{array}$$

(6.) Divide $x^3+8y^3-27z^3+18xyz$ by $x+2y-3z$.

$$
\begin{array}{l}
x+2y-3z\,\big)\,x^3+18xyz+8y^3-27z^3\,\big(x^2-2xy+3xz+4y^2+6yz+9z^2 \\
\quad x^3+2x^2y-3x^2z \\
\quad\overline{\quad -2x^2y+3x^2z+18xyz+8y^3-27z^3} \\
\quad -2x^2y-4xy^2+6xyz \\
\quad\overline{\qquad 3x^2z+4xy^2+12xyz+8y^3-27z^3} \\
\qquad 3x^2z+6xyz-9xz^2 \\
\qquad\overline{\qquad\quad 4xy^2+6xyz+9xz^2+8y^3-27z^3} \\
\qquad\quad 4xy^2+8y^3\quad -12y^2z \\
\qquad\quad\overline{\qquad\quad 6xyz+12y^2z+9xz^2-27z^3} \\
\qquad\quad 6xyz+12y^2z-18yz^2 \\
\qquad\quad\overline{\qquad\quad\quad 9xz^2+18yz^2-27z^3} \\
\qquad\quad\quad 9xz^2+18yz^2-27z^3
\end{array}
$$

It is particularly necessary in such a sum as this to remember that the same order of the letters and their powers must be maintained throughout the whole work.

By applying some of the theorems in Art. 57, the above division may be considerably shortened.

$$
\begin{array}{l|l}
(x+2y)-3z\,\big|\,(x^3+8y^3)+18xyz-27z^3 & (x^2-2xy+4y^2) \\
\quad\big|\,\overline{(x^3+8y^3)-3(x^3-2xy+4y^2)z} & \\
\qquad 3(x^2+4xy+4y^2)z-27z^3 & \\
\qquad \text{Or,}\quad 3(x+2y)^2z-27z^3 & +3(x+2y)z \\
\qquad\quad\overline{3(x+2y)^2z-9(x+2y)z^2} & \\
\qquad\quad 9(x+2y)z^2-27z^3 & +9z^2 \\
\qquad\quad 9(x+2y)z^2-27z^3 &
\end{array}
$$

The same answer as before, in different order.

(7.) Divide $x^3-(2a+b)x^2+(a^2+2ab)x-a^2b$ by $x-b$.

$$
\begin{array}{l}
x-b\,\big)\,x^3-(2a+b)x^2+(a^2+2ab)x-a^2b\,\big(x^2-2ax+a^2 \\
\quad x^3-bx^2 \\
\quad\overline{\quad -2ax^2+(a^2+2ab)x} \\
\quad -2ax^2+2abx \\
\quad\overline{\qquad\quad a^2x-a^2b} \\
\qquad\quad a^2x-a^2b
\end{array}
$$

(8.) $x^5+mx^4y-(n+1)x^3y^2-(m+1)x^2y^3+nxy^4+y^5$ by $x+y$.

$$
\begin{array}{l|l|l}
x+y & x^5+mx^4y-(n+1)x^3y^2-(m+1)x^2y^3+nxy^4+y^5 & x^4+(m-1)x^3y \\
 & x^5+x^4y & -(m+n)x^2y^2 \\
 & \hline (m-1)x^4y-(n+1)x^3y^2 & +(n-1)xy^3 \\
 & (m-1)x^4y+(m-1)x^3y^2 & +y^4 \\
 & \hline -(m+n)x^3y^2-(m+1)x^2y^3 & \\
 & -(m+n)x^3y^2-(m+n)x^2y^3 & \\
 & \hline (n-1)x^2y^3+nxy^4 & \\
 & (n-1)x^2y^3+(n-1)xy^4 & \\
 & \hline xy^4+y^5 & \\
 & xy^4+y^5 &
\end{array}
$$

EXAMPLES FOR PRACTICE.—XVII.

(1.) Divide $18a^3b^2c^5$ by $9ab^2c^3$.
(2.) Divide $-20a^4xy$ by $5ay^2$.
(3.) Divide $6a^5b^2c^3$ by $-4a^2b^2c^2$.
(4.) Divide $-56a^2xy^3$ by $-63a^3x^4y^5$.
(5.) Divide $12a^3-18a^2x+24ax^2$ by 6.
(6.) Divide $12a^3-18a^2x+24ax^2$ by $8a$.
(7.) Divide $6ax^3+9a^2x^2-12a^3x$ by $9a^2x^2$.
(8.) Divide $-7abc-8bcd+9acd+10abd$ by $-abcd$.
(9.) Divide $a^3m^2x^2-4a^2mxy+6ay^2$ by $-3a^2m$.
(10.) Divide x^2-3x-4 by $x-4$.
(11.) Divide $15x^2+34xy+15y^2$ by $3x+5y$.
(12.) Divide $4x^3-15x^2y+10xy^2-3y^3$ by $x-3y$.
(13.) Divide $a^6-11a^2x^4-6x^6$ by a^2+3x^2.
(14.) Divide $3x^4-7x^3-4x^2+5x-1$ by x^2-3x+1.
(15.) Divide $x^5-2x^3-2x^2-3x-2$ by x^2+2x+1.
(16.) Divide $6x^2-5x-3$ by $2x-3$.
(17.) Divide $4x^5+7x^3+4x$ by $2x^2+x+1$.
(18.) Divide $x^4+x^3-10x^2+5$ by x^3-2x^2-2x+1.
(19.) Divide $x^2-4xz-4y^2-4yz+3z^2$ by $x+2y-z$.
(20.) Divide $8x^3+y^3+6xy-1$ by $2x+y-1$

(21.) Divide $x^3 + y^3 - z^3 - 1 + 3(x^2y + xy^2 - z^2 - z)$ by $x + y - z - 1$.

(22.) Divide 1 by $1 + x^2$.

(23.) Divide $1 + x - x^2$ by $1 - x + x^2$.

(24.) Divide $(1 - x^2)^3$ by $(1 - x^3)^2$.

(25.) Divide $x^3 - (a - b)x^2 + ab^2$ by $x + b$.

(26.) Divide $x^4 - (a + b)x^3 + (a + ab + b)x^2 - (a^2 + b^2)x + ab$ by $x^2 - ax + b$.

(27.) Divide $x^4 - 2bx^3 - (a^2 - b^2)x^2 + 2a^2bx - a^2b^2$ by $x^2 - (a + b)x + ab$.

(28.) Divide $1 - 2bx - (a^2 - 2b - b^2)x^2 - 2(a^2 + b^2)x^3 - (a^2 - 2a^2b - b^2)x^4 + 2a^2bx^5 - a^2b^2x^6$ by $1 - (a + b)x - (a - b)x^2 + abx^3$.

(29.) Divide $1 + ax + bx^2 + (1 + a + c)x^3 + bx^4 + (1 + a + c)x^5 + bx^6 + cx^7 + x^8$ by $1 - x + x^2$.

(30.) Divide $a + bx + cx^2 + dx^3$ by $1 + x + x^2$ to six terms.

73. The tedious process that is necessary in long division may often be shortened or altogether avoided by the use of the theorems stated in Art. 57.

If $(a + b)(a - b) = a^2 - b^2$, it is evident that $\frac{a^2 - b^2}{a - b} = a + b$, and $\frac{a^2 - b^2}{a + b} = a - b$.

Whenever, then, the divisor can be recognized as a factor of the dividend, it should be struck out, and the other factor at once written down as the quotient.

Illustrative Examples.

(1.) Divide $x^6 + 8y^3$ by $x^2 + 2y$.

By Theorem IV. the factors of $(x^2)^3 + (2y)^3$ are $x^2 + 2y$, and $(x^2)^2 - x^2 \cdot 2y + (2y)^2$.

$$\therefore \frac{(x^2)^3 + (2y)^3}{x^2 + 2y} = (x^2)^2 - x^2 \cdot 2y + (2y)^2 = x^4 - 2x^2y + 4y^2.$$

(2.) Divide $x^4 + 9x^2 + 81$ by $x^2 + 3x + 9$.

By Theorem VI. the factors of $x^4 + 9x^2 + 81$ are $x^2 + 3x + 9$ and $x^2 - 3x + 9$.

$$\therefore \frac{x^4 + 9x^2 + 81}{x^2 + 3x + 9} = x^2 - 3x + 9.$$

74. Factoring.—To help the student to acquire facility in recognizing the factors of a product—a thing necessary to a good algebraist—some examples in factoring are here subjoined.

(1.) Resolve $a^4 - a^2b^2$ into simple factors.

As a^2 is common to both terms, divide it out; the quotient is $a^2 - b^2$, which, by Theorem III., is equal to $(a + b)(a - b)$.

$$\therefore a^4 - a^2b^2 = a^2(a + b)(a - b).$$

(2.) Resolve $x^2 + 5x + 6$ into simple factors.

By Theorem X. $(x + a)(x + b) = x^2 + (a + b)x + ab$.

It is here to be observed that when $x + a$ is multiplied by $x + b$, the first term of the product is x^2, the second is x with the *sum* of a and b for a coefficient, and the third is the *product* of a and b. When, therefore, a product of the form $x^2 + mx + n$ is given to be resolved into factors, we first find the factors of the third term, and then ascertain if their sum amounts to the coefficient of x.

In the above question, the factors of 6 are 6 and 1, or 3 and 2; the sum of 6 and 1 is 7, of 3 and 2 is 5; and as this latter number is the coefficient of x, we conclude that the required factors are $x + 2$ and $x + 3$.

$$\text{Or, } x^2 + 5x + 6 = (x + 2)(x + 3)$$

In the same way, $x^2 + 24x + 119 = (x + 7)(x + 17)$.

(3.) Find the simple factors of $x^2 - 7x + 12$.

This is of the form indicated by the lower sign in Theorem X., and the factors must be of the form $x - a$, $x - b$.

First, we resolve 12 into pairs of factors in every possible way, thus: $12 = 12 \times 1$, or 6×2, or 3×4.

As 3×4 is the only pair whose sum is 7, the factors are $x - 3$ and $x - 4$.

And, $x^2 - 7x + 12 = (x - 3)(x - 4)$.

Similarly, $x^2 - 17x + 72 = (x - 8)(x - 9)$.

(4.) Resolve $x^2 - 7x - 18$ into simple factors.

This is of the form indicated by the lower sign in Theorem XI., the coefficient of x being the difference between a and b, and the factors take the form of $x - a$, $x + b$.

First, resolving 18 into every possible pair of factors—

$$18 = 18 \times 1, \text{ or } 9 \times 2, \text{ or } 6 \times 3.$$

Here 9 and 2, being the only pair whose difference is 7, must be the pair required; and as this difference is minus, the 9 must be minus. The factors, therefore, of $x^2 - 7x - 18$ are $x - 9$ and $x + 2$.

So also the factors of $x^2 + x - 72$ are found to be $x + 9$ and $x - 8$.

75. The following general rules for resolving products of the forms $x^2 + mx + n$, $x^2 - mx + n$, $x^2 - mx - n$, and $x^2 + mx - n$, into their simple factors, may be found useful:—

(a) If the sign of the third term (n) is plus, the signs of the second terms (a and b) of the factors are alike, either both plus or both minus.

(b) In that case, the coefficient (m) of the second term of the product is the *sum* of the second terms of the factors; and the sign of these terms is the same as that of the second term of the product.

(c) If the sign of the third term is minus, the signs of the second terms of the factors are unlike, one plus, the other minus.

(d) In that case, the coefficient of the second term of the product is the *difference* between the second terms of the factors; and the signs of these terms being unlike, the greater will have the sign of the second term of the product.

If the coefficient of the second term is not formed by the sum of the factors of the third when the third is positive, or by their difference when the third is negative, then the expression cannot be resolved into factors.

$x^2-7x+36$ and $x^2-8x-24$ have no factors; for the factors of 36 being 36×1, 18×2, 12×3, 9×4, and 6×6, 7 is not the sum of the numbers in any one of the pairs, and the factors of 24 being 24×1, 12×2, 8×3, and 6×4, the differences are 23, 10, 5, and 2, none of which is the coefficient of the second term x.

(5.) Find the simple factors of x^6-64y^6.

By Theorem III. $x^6-64y^6=(x^3+8y^3)(x^3-8y^3)$.

By Theorem IV. $x^3+8y^3=(x+2y)(x^2-2xy+4y^2)$.

By Theorem V. $x^3-8y^3=(x-2y)(x^2+2xy+4y^2)$.

$\therefore\ x^6-64y^6=(x+2y)(x-2y)(x^2-2xy+4y^2)(x^2+2xy+4y^2)$.

(6.) Resolve into elementary factors $ac-2bc-3ad+6bd$. From the first two, taking out the common factor c, and from the last two, the common factor $3d$, we have

$$(a-2b)c-3d(a-2b);$$

and from these, taking out the common factor $a-2b$, we have

$$(a-2b)(c-3d).$$

$$\therefore\ ac-2bc-3ad+6bd=(a-2b)(c-3d).$$

This may also be resolved by taking out the common factor from the first and third, and from the second and fourth, thus: $\overbrace{ac-3ad}+\underbrace{6bd-2bc}=a(c-3d)-2b(c-3d)$

$$=(a-2b)(c-3d).$$

(7.) Find the simple factors of $acx^2-(3ad+2bc)x+6bd$.

$$acx^2-3adx-2bcx+6bd=a(cx-3d)x-2b(cx-3d)$$

$$=(ax-2b)(cx-3d).$$

In the same way the simple factors of $12x^2 - 31x + 20$ may be found.

$$\begin{aligned} 12x^2 - 31x + 20 &= 12x^2 - 16x - 15x + 20, \\ &= 4x(3x - 4) - 5(3x - 4), \\ &= (4x - 5)(3x - 4). \end{aligned}$$

(8.) What are the simple factors of $x^3 - 7x - 6$?

In this question we take one of the factors of 6, say 2, and suppose $x + 2$ or $x - 2$ to be a factor of the given expression. Let us take $x + 2$: then—

$$\begin{aligned} x^3 - 7x - 6 &= x^3 + 2x^2 - 2x^2 - 4x - 3x - 6, \\ &= x^2(x + 2) - 2x(x + 2) - 3(x + 2), \\ &= (x^2 - 2x - 3)(x + 2), \\ &= (x - 3)(x + 1)(x + 2). \end{aligned}$$

$+ 2x^2$ is inserted in the first line in order to make the first pair of terms divisible by $x + 2$, and $- 2x^2$ is introduced to prevent any change in the value of the expression. Whatever is added to a line must be taken away again.

Let us suppose $x + 3$ to be one of the factors, then $x^3 - 7x - 6 = x^3 + 3x^2 - 3x^2 - 9x + 2x - 6 = x^2(x + 3) - 3x(x + 3) + 2(x - 3)$, from which we see at once that $x + 3$ is not a factor, as the third pair of terms is not divisible by it.

But if $x - 3$ be supposed a factor, then—

$$\begin{aligned} x^3 - 7x - 6 &= x^3 - 3x^2 + 3x^2 - 9x + 2x - 6, \\ &= x^2(x - 3) + 3x(x - 3) + 2(x - 3), \\ &= (x^2 + 3x + 2)(x - 3), \\ &= (x + 1)(x + 2)(x - 3). \end{aligned}$$

(9.) Resolve $a^2 - b^2 + 2bc - c^2$ into simple factors.

$$\begin{aligned} a^2 - b^2 + 2bc - c^2 &= a^2 - (b^2 - 2bc + c^2), \\ &= a^2 - (b - c)^2, \\ &= (a + b - c)(a - b + c). \end{aligned}$$

(10.) What are the simple factors of

$$4(xy + z)^2 - (x^2 + y^2 - z^2 - 1)^2?$$

$4(xy+z)^2-(x^2+y^2-z^2-1)^2$
$=\{2(xy+z)+x^2+y^2-z^2-1\}\{2(xy+z)-x^2-y^2+z^2+1\},$
$=\{x^2+2xy+y^2-z^2+2z-1\}\{z^2+2z+1-x^2+2xy-y^2\},$
$=\{(x+y)^2-(z-1)^2\}\{(z+1)^2-(x-y)^2\},$
$=(x+y+z-1)(x+y-z+1)(x-y+z+1)(y-x+z+1).$

(11.) Find the simple factors of

$$(x-2)^2(y^2-4)-(y-2)^2(x^2-4).$$

$(x-2)^2(y^2-4)-(y-2)^2(x^2-4),$
$=(x-2)(y-2)\{(x-2)(y+2)-(y-2)(x+2)\},$
$=(x-2)(y-2)(xy+2x-2y-4-xy+2x-2y+4),$
$=4(x-2)(y-2)(x-y).$

EXAMPLES FOR PRACTICE.—XVIII.

Resolve the following into elementary factors:—

(1.) $a^3bc+ab^2c^2$.
(2.) $a^2x^2-y^2$.
(3.) a^3bx-ab^3x.
(4.) $5a^3x-20ax^3$.
(5.) $4a^2+8a+4$.
(6.) $a-6a^2+9a^3$.
(7.) $x(2x+1)^2-x$.
(8.) $(x+y)^2-z^2$.
(9.) $x^2-(y-z)^2$.
(10.) $(a-2b)^2-9c^2$.
(11.) $(m+p)^2-(n+q)^2$.
(12.) $(a^2+b^2)^2-4a^2b^2$.
(13.) $x^2+7x+12$.
(14.) $x^2-7x+10$.
(15.) $x^2+3x-10$.
(16.) x^2-3x-4.
(17.) $x^2+13x+12$.
(18.) x^2+x-12.
(19.) $x^2+17x+60$.
(20.) $x^2-17x-60$.
(21.) $x^2-(a+c)x+ac$.
(22.) $x^2-(a-b)x-ab$.
(23.) $a^2x^2-a(b+c)xy+bcy^2$.
(24.) $acx^2-(ad-bc)xy-bdy^2$.
(25.) $12x^2-25xy+12y^2$.
(26.) $24x^2-2x-15$.
(27.) $x^2+xy+x+y$.
(28.) $x^2+y^2+2xy+x+y$.
(29.) x^3-x^2-3x-1.
(30.) $x^3+x^2-8x-12$.
(31.) $4x^2-9y^2+24yz-16z^2$.
(32.) $(x^3-x)^2-(x-1)^2$.
(33.) $8a^3+27x^3$.
(34.) x^6+y^6.
(35.) x^6-y^6.
(36.) $16a^4-250ax^3$.
(37.) $x(x^3+y^3)+x^3+y^3$.

(38.) $2^6x^3 - 1$.
(39.) $a^3x^6 + b^9y^3$.
(40.) $x^4y^4 - x^4 - y^4 + 1$.
(41.) $16 + 4x^2 + x^4$.
(42.) $a^4b^4 + 9a^2b^2 + 81$.
(43.) $x^8 + x^4y^4 + y^8$.
(44.) $x^4 - 16$.
(45.) $x^4 + 4$.
(46.) $x^8 - 16$.
(47.) $(x+y)^3 - xz^2 - yz^2$.
(48.) $4x^4 - 32xy^3 + x^3 - 8y^3$.
(49.) $x^2(x^3 - y^3) - x^3y^2 + y^5$.
(50.) $(x-1)^2(y^2-1) - (y-1)^2(x^2-1)$.

Perform the operations indicated in the following :—

$$\frac{x^2 - 8y + 2xy - 4x}{x-4} + \frac{x^3 - 1 - (x-1)^3}{x-1} - \frac{x^3 - x^2y + xy^2 - y^3}{x^2+y^2}.$$

Arrange and resolve into factors. Then :—

$$\frac{x^2+2xy-4x-8y}{x-4} + \frac{x^3-1-x^3+3x^2-3x+1}{x-1} - \frac{x^3-x^2y+xy^2-y^3}{x^2+y^2},$$

$$= \frac{x(x+2y) - 4(x+2y)}{x-4} + \frac{3x^2-3x}{x-1} - \frac{x^2(x-y)+y^2(x-y)}{x^2+y^2},$$

$$= \frac{(x-4)(x+2y)}{x-4} + \frac{3x(x-1)}{x-1} - \frac{(x^2+y^2)(x-y)}{x^2+y^2},$$

$$= x + 2y + 3x - x + y = 3(x+y).$$

EXAMPLES FOR PRACTICE.—XIX.

(1.) $\dfrac{x^4 - 2x^2 + 1}{x^2 - 1} + \dfrac{x^4 + 2x^2 + 1}{x^2 + 1}$.

(2.) $\dfrac{x^3 - 27}{x^2 + 3x + 9} - \dfrac{x^3 - 64}{x^2 + 4x + 16}$.

(3.) $\dfrac{x^4 + 4}{x^2 + 2x + 2} + \dfrac{x^4 - 16}{x^2 + 4}$.

(4.) $\dfrac{(x+a)^2 - (x-b)^2}{2x + (a-b)} + \dfrac{(x-b)^2 - (x-c)^2}{2x - (b+c)}$.

(5.) $\dfrac{(2x+y)^3-1}{2x+y-1} - \dfrac{(2x-y)^3+1}{2x-y+1}$.

(6.) $\dfrac{x^4+4x^2+16}{x^2+2x+4} + \dfrac{x^4+2x^3+8x-16}{x^2+4}$.

76. Synthetical Division.—As in multiplication, the work in division may often be shortened by using the coefficients alone. Thus:—

Divide $a^4 - a^3b - 5a^2b^2 + 7ab^3 - 2b^4$ by $a^2 + 2ab - b^2$.

$$
\begin{array}{l}
1+2-1 \,)\, 1-1-5+7-2 \,(\, 1-3+2 \\
\quad \underline{1+2-1} \quad \text{(A)} \\
\quad\quad -3-4+7 \\
\quad\quad \underline{-3-6+3} \quad \text{(B)} \\
\quad\quad\quad\quad 2+4-2 \\
\quad\quad\quad\quad 2+4-2 \quad \text{(C)}
\end{array}
$$

$a^4 \div a^2$ gives a^2, and as the powers of a are in descending order, and of b in ascending, the quotient is readily written down— $a^2 - 3ab + 2b^2$.

In this process the several partial products marked (A), (B), (C), with the remainder, if there be one, are together equal to the dividend; and the several coefficients of the quotient are respectively the same as those of the first terms of the partial products.

If, then, we add these partial products, *with their signs changed*, to the dividend, the sum will be nothing; but if we do the same, leaving out the first term of each, the sum will be the coefficients of the quotient. Thus—

$$
\begin{array}{r}
1-1-5+7-2 \\
-2+1 \\
\quad +6-3 \\
\quad\quad -4+2 \\
\hline
1-3+2
\end{array}
$$

77. The work is generally performed as follows:—

The coefficients of the divisor are written vertically in front with all their signs, except the first, changed.

The first term of the dividend is then written under itself, and used as a multiplier to all the terms of the divisor except the first.

In the above example, the first term being 1, we say once -2 is -2, and place the product under the second term of the dividend; once $+1$ is $+1$, which we put under the third term.

After the two quantities in the second column have been added together, the work will stand thus:—

$$\begin{array}{r|l} 1 & 1-1-5+7-2 \\ -2 & \quad -2 \\ +1 & \quad\quad +1 \\ \hline & 1-3 \end{array}$$

We now use this -3 as a multiplier, the first product, $+6$, being placed under -5, and the second under $+7$, each opposite the figure of the divisor from which it has been derived.

When the third column has been added up, we will have—

$$\begin{array}{r|l} 1 & 1-1-5+7-2 \\ -2 & \quad -2+6 \\ +1 & \quad\quad +1-3 \\ \hline & 1-3+2 \end{array}$$

Now using $+2$ as a multiplier, the next step will give

$$\begin{array}{r|l} 1 & 1-1-5+7-2 \\ -2 & \quad -2+6-4 \\ +1 & \quad\quad +1-3+2 \\ \hline & 1-3+2+0 \end{array}$$

ork continuing until it extends to the last term of dividend.

Illustrative Examples.

Divide $4x^5 + 13x^4 + 2x^3 - 2x^2 + 2x - 3$ by $x^2 + 4x + 3$.

$$\begin{array}{r|l} 1 & 4+13+\ 2-2+2-3 \\ -4 & \quad -16+12-8+4 \\ -3 & \qquad\quad -12+9-6+3 \\ \hline & 4-\ 3+\ 2-1+0+0 \end{array}$$

$\therefore$ *Ans.* $= 4x^3 - 3x^2 + 2x - 1$.

Divide $x^6 - x^5 + x^4 + 2x^2 + 3x - 6$ by $x^3 + x - 2$.

$$\begin{array}{r|l} 1 & 1-1+1+0+2+3-6 \\ +0 & \quad 0-0+0+0 \\ -1 & \qquad -1+1-0-3 \\ +2 & \qquad\quad +2-2+0+6 \\ \hline & 1-1+0+3 \end{array}$$

$$\begin{array}{r|l} 1 & 1-1+1+0+2+3-6 \\ -1 & \qquad -1+1-0-3 \\ +2 & \qquad\quad +2-2+0+6 \\ \hline & 1-1+0+3 \end{array}$$

$\therefore$ *Ans.* $= x^3 - x^2 + 3$.

3. When the coefficient of the first term of the divisor [illegible] 1, the sums of the several columns must [illegible] before using them as multipliers, since [illegible]s of the coefficients of the quotient by [illegible]he divisor.

[illegible]$^6 + 8x^5 - 7x^4 - 2x^2 - 16x - 5$ by $3x^2 + 4x + 1$.

$$\begin{array}{r|l} & 6+8-7+\ 0-\ 2-16-5 \\ & \quad -8+0+12-16+20 \\ & \qquad -2+\ 0+\ 3-\ 4+5 \\ \hline 3) & 6+0-9+12-15 \\ \hline & 2+0-3+\ 4-\ 5 \end{array}$$

$\therefore$ *Ans.* $= 2x^4 - 3x^2 + 4x - 5$.

(4.) Divide $2x^4 - 7x^3 - 4x^2 + 3x + 2$ by $2x^2 + 3x - 1$.

$$\begin{array}{r|l} 2 & 2 - 7 - 4 + 3 + 2 \\ -3 & \quad - 3 + 15 - 18 \\ +1 & \quad\quad + 1 - 5 + 6 \\ \hline 2) & 2 - 10 + 12 - 20 + 8 \\ & \underline{1 - 5 + 6} \end{array}$$

Here the sum of the last two columns does not become 0, as it did in the previous examples; but as they correspond to the last two terms of the dividend, neither of which contains a power of x large enough to be divided by the x^2 of the divisor, no further division is possible: $-20+8$, therefore, represents a remainder, and as -20 stands under the column corresponding to $3x$ of the original dividend, the remainder is $-20x+8$.

$\therefore$ The answer is $x^2 - 5x + 6 - \dfrac{20x - 8}{2x^2 + 3x - 1}$. (Art. 27.)

In addition to the following, many of the examples in Art. 72 may be solved by this method, which is called Synthetical Division.

EXAMPLES FOR PRACTICE.—XX.

(1.) Divide $x^6 - 8x^4 - x^2 - 28x + 12$ by $x^3 + 4x^2 + 3x - 2$.

(2.) Divide $4a^{10}b^5 - 8a^8b^4 + 16a^2b - 32$ by $a^4b^2 - 2a^2b + 2$.

(3.) Divide $9 + 14x^4 - 23x^8 - 36x^{12}$ by $3 - 4x^2 + 5x^4 - 6x^6$.

(4.) Divide $4 + 13x + 2x^4 + 2x^6 + x^7$ by $4 - 3x + x^3$.

(5.) Divide $4x^5 - 9x^4 - 11x^3 - 10x^2 - 14x + 3$ by $4x^2 + 3x - 2$.

(6.) Divide $50x^7 - 25x^6y + 30x^5y^2 + y^7$ by $5x^3 + 3xy^2 - y^3$.

79. Literal Exponents.—We have hitherto used numbers only as exponents, but in many investigations it is necessary to use letters, or a combination of letters and numbers.

The addition, subtraction, multiplication, and division

of such quantities, however, are performed in the same manner as when the exponents are numbers alone.

(1.) Add $3x^m$, $2x^n$, $5x^m$, and $-4x^n$.

Ans. $8x^m - 2x^n$.

(2.) Subtract $3x^{2m} - 4x^{m+n} + 5x^{2n}$ from $5x^{2m} - 2x^{m+n} + 3x^{2n}$.

Ans. $2x^{2m} + 2x^{m+n} - 2x^{2n}$.

(3.) Multiply $4x^m y^p$ by $3x^n y^q$.

Ans. $12x^{m+n} y^{p+q}$.

(4.) Multiply $x^{3n} - 3x^{2n} + 3x^n - 1$ by $x^{2n} - 2x^n + 1$.

$$\begin{array}{r} x^{3n} - 3x^{2n} + 3x^n - 1 \\ x^{2n} - 2x^n + 1 \\ \hline x^{5n} - 3x^{4n} + 3x^{3n} - x^{2n} \\ - 2x^{4n} + 6x^{3n} - 6x^{2n} + 2x^n \\ x^{3n} - 3x^{2n} + 3x^n - 1 \\ \hline x^{5n} - 5x^{4n} + 10x^{3n} - 10x^{2n} + 5x^n - 1 \end{array}$$

(5.) Multiply $a^n - na^{n-1} + n(n-1)a^{n-2}$ by $a^2 + a + 1$.

$$\begin{array}{l} a^n - na^{n-1} + n(n-1)a^{n-2} \\ a^2 + a + 1 \\ \hline a^{n+2} - na^{n+1} + n(n-1)a^n \\ \quad a^{n+1} - na^n + n(n-1)a^{n-1} \\ \quad\quad a^n - na^{n-1} + n(n-1)a^{n-2} \\ \hline a^{n+2} - (n-1)a^{n+1} + (n-1)^2 a^n + n(n-2)a^{n-1} + n(n-1)a^{n-2} \end{array}$$

(6.) Multiply $a^n - a^{n-1}x + a^{n-2}x^2 - \ldots + ax^{n-1} - x^n$ by $a + x$.

$$\begin{array}{l} a^n - a^{n-1}x + a^{n-2}x^2 - \ldots + ax^{n-1} - x^n \\ a + x \\ \hline a^{n+1} - a^n x + a^{n-1}x^2 - \ldots + a^2 x^{n-1} - ax^n \\ \quad + a^n x - a^{n-1}x^2 + \ldots - a^2 x^{n-1} + ax^n - x^{n+1} \\ \hline a^{n+1} \qquad\qquad\qquad\qquad\qquad\qquad - x^{n+1} \end{array}$$

(7.) Divide x^m by x^n.

By Art. 65, $\frac{x^m}{x^n} = x^{m-n}$ or $\frac{1}{x^{n-m}}$, according as m is greater or less than n.

(8.) Divide $a^{x+1}+a^{x-2}$ by a^2+a.

$$
\begin{array}{r|l}
a^2+a & a^{x+1}+a^{x-2} \;(\; a^{x-1}-a^{x-2}+a^{x-3} \\
 & \underline{a^{x+1}+a^{x}} \\
 & \quad -a^{x}+a^{x-2} \\
 & \quad \underline{-a^{x}-a^{x-1}} \\
 & \qquad\quad a^{x-1}+a^{x-2} \\
 & \qquad\quad a^{x-1}+a^{x-2}
\end{array}
$$

EXAMPLES FOR PRACTICE.—XXI.

(1.) Multiply x^m by x^n, and $x^m y$ by xy^n.

(2.) Multiply a^x, a^y, and a^z together.

(3.) Multiply $a^{2m}+x^{2n}$ by a^m-x^n.

(4.) Multiply $x^{m+1}-x^m y+xy^m-y^{m+1}$ by $x+y$.

(5.) Multiply $ax^{3m}-bx^{2m}+cx^m-dx$ by $ax^{2m}+bx^m-cx$.

(6.) Multiply $x^{m-1}+(m-1)x^{m-2}+(m-1)^2x^{m-3}+\ldots\ldots+(m-1)^{m-2}x+(m-1)^{m-1}$ by $x-m+1$.

(7.) Divide $x^{2m+2}-x^{2m+1}+x^{2m}$ by x^{m+1}.

(8.) Divide $x^{3m}y^{2m+1}+x^{2m+1}y^{3m}$ by $x^{2m-1}y^{m+1}$.

(9.) Divide $a^{2m}-2a^m x^n+x^{2n}$ by a^m-x^n.

(10.) Divide $a^{n+1}+a^{n-2}$ by $a+1$.

(11.) Divide $a^2x^{3m}+(2ac-b^2)x^{2m-2}+c^2x^{2m-4}$ by $ax^m+bx^{m-1}+cx^{m-2}$.

(12.) Divide $x^{n+2}+n^2(n-2)x^{n-1}+n^2(n-1)x^{n-2}$ by x^2+nx+n.

80. Negative Exponents.—In dividing x^m by x^n, when m is greater than n, we have the quotient x^{m-n} (Art. 65).

If, after performing the division on the supposition that m is greater than n, it should turn out that m is less than n, the quotient x^{m-n} would have a negative exponent, as x^{-r}.

By Art. 64 we know that $\dfrac{x^m}{x^m}=1$; and by Art. 65 that $\dfrac{x^m}{x^m}=x^{m-m}=x^0$. $\therefore x^0=1$.

Multiplying x^{-r} by x^r, we have, by Art. 79, or Art. 49,

$$x^{-r}x^r = r^{r-r} = x^0 = 1;$$

$$\text{And} \quad \therefore \quad x^{-r} = \frac{1}{x^r}.$$

Or, x^{-r} is the reciprocal of x^r.

The addition, subtraction, multiplication, and division of such quantities are performed in the same manner as when the exponents are positive.

The following example will sufficiently illustrate this:—

Divide $a^{-5} - 5a^{-1}x^4 - 4x^5$ by $a^{-2} + 2a^{-1}x + x^2$.

$$\begin{array}{r|l}
a^{-2}+2a^{-1}x+x^2 & a^{-5}-5a^{-1}x^4-4x^5 \\
\cline{2-2}
 & a^{-3}-2a^{-2}x+3a^{-1}x^2-4x^3 \\
 & a^{-5}+2a^{-4}x+a^{-3}x^2 \\
\cline{2-2}
 & -2a^{-4}x-a^{-3}x^2-5a^{-1}x^4-4x^5 \\
 & -2a^{-4}x-4a^{-3}x^2-2a^{-2}x^3 \\
\cline{2-2}
 & 3a^{-3}x^2+2a^{-2}x^3-5a^{-1}x^4-4x^5 \\
 & 3a^{-3}x^2+6a^{-2}x^3+3a^{-1}x^4 \\
\cline{2-2}
 & -4a^{-2}x^3-8a^{-1}x^4-4x^5 \\
 & -4a^{-2}x^3-8a^{-1}x^4-4x^5
\end{array}$$

EXAMPLES FOR PRACTICE.—XXII.

(1.) Divide $a^{-2} - x^{-2}$ by $a^{-1} - x^{-1}$.

(2.) Divide $x - x^{-3}a^4$ by $1 + x^{-1}a$.

(3.) Divide $x + a + a^5x^{-4} + a^6x^{-5}$ by $x^{-2} + 2ax^{-3} + a^2x^{-4}$.

(4.) Divide $x^{-2} - a^2 + (2ab - c)x - (ac + b^2 - d)x^2 + (ad + bc)x^3 - bdx^4$ by $x^2 + ax^3 - bx^4$.

(5.) Divide $a^{-(m-2)} + a^{-m} + a^{-(m+2)}$ by $a^2 - a + 1$.

(6.) Divide x^{3n} by $x^n - 1$.

Substitutions.—Find the value of the following when $a = 1$, $b = 2$, $c = 3$, $d = 0$, $x = 5$, $y = 4$, $z = 3$, $m = 4$, and $n = 2$.

(1.) $\dfrac{a^2b^3xy^4}{b^2cy^2z} = \dfrac{a^2bxy^2}{cz} = \dfrac{1 \times 2 \times 5 \times 16}{3 \times 3} = \frac{160}{9} = 17\frac{7}{9}$.

(2.) $\dfrac{(a^3-b^3)(b^3-c^3)(c^3-a^3)}{(a^2+ab+b^2)(b^4+c)(c^2+ac+a^2)} = \dfrac{(a-b)(b^3-c^3)(c-a)}{b^4+c}$

$\dfrac{(-1)\times(-19)\times 2}{16+3} = \dfrac{19\times 2}{19} = 2.$

(3.) $x^{-m}+2ax^{-n}+a^2 = \dfrac{1}{5^4}+\dfrac{2\times 1}{5^2}+1 = 1+\dfrac{1}{5^4}(1+50) = 1\frac{51}{625}.$

EXAMPLES FOR PRACTICE.—XXIII.

Find the values of the following expressions, a, b, c, etc., standing for the same numbers as above :—

(1.) $\dfrac{abc}{xy}-\dfrac{bcd}{yz}+\dfrac{cda}{zx}$.

(2.) $\dfrac{x^2-xy+y^2}{x^3+y^3}+\dfrac{a^2-ab+b^2}{a^3+b^3}$.

(3.) $\dfrac{x^a-y^b+z^c}{a^m+b^n+c^d}$.

(4.) $x^{-m}+y^{-n}$.

(5.) $\dfrac{a^{-m}+a^{-(m+1)}+a^{-(m+2)}}{a^2-a+1}$.

(6.) $\dfrac{x^{-b}-y^{-c}+z^{-d}}{x^{-d}-y^{-a}+z^{-b}}\times\dfrac{b^{-2}+c^{-3}}{6^{-m}+7^{-s}}$.

81. Theorems in Division. — The following theorems will be found useful in factoring :—

Divide x^n-a^n by $x-a$.

$$
\begin{array}{l|l}
x-a)\,x^n-a^n & (x^{n-1}+ax^{n-2}+a^2x^{n-3}+a^3x^{n-4}+\text{etc.}\\
\quad x^n-ax^{n-1} & \\
\hline
\quad\quad ax^{n-1}-a^n & \\
\quad\quad ax^{n-1}-a^2x^{n-2} & \\
\hline
\quad\quad\quad a^2x^{n-2}-a^n & \\
\quad\quad\quad a^2x^{n-2}-a^3x^{n-3} & \\
\hline
\quad\quad\quad\quad a^3x^{n-3}-a^n & \\
\quad\quad\quad\quad a^3x^{n-3}-a^4x^{n-4} & \\
\hline
\quad\quad\quad\quad\quad a^4x^{n-4}-a^n &
\end{array}
$$

Here, if n be *one*, the quotient will be x^0 or 1 (Art. 80), and the remainder $ax^0 - a$ or 0; if n be *two*, the quotient will be $x + a$, and the remainder $a^2x^0 - a^2$ or 0; if n be *three*, the quotient will become $x^2 + ax + a^2$, and the remainder $a^3x^0 - a^3$ or 0; if n be *four*, the quotient will be $x^3 + ax^2 + a^2x + a^3$, and the remainder $a^4x^0 - a^4$ or 0; and so on for higher numbers.

From which we have—

$$\text{(I.)}\quad \frac{x^n - a^n}{x - a} = x^{n-1} + ax^{n-2} + a^2x^{n-3} + a^3x^{n-4} +, \text{ etc.},$$

with no remainder, whatever may be the value of n.

In the same way the following can also be shown :—

$$\text{(II.)}\quad \frac{x^n + a^n}{x + a} = x^{n-1} - ax^{n-2} + a^2x^{n-3} - a^3x^{n-4} +, \text{ etc.},$$

with a remainder of $2a^n$ when n is even, and none when n is odd.

$$\text{(III.)}\quad \frac{x^n - a^n}{x + a} = x^{n-1} - ax^{n-2} + a^2x^{n-3} - a^3x^{n-4} +, \text{ etc.},$$

with a remainder of $-2a^n$ when n is odd, and none when n is even.

$$\text{(IV.)}\quad \frac{x^n + a^n}{x - a} = x^{n-1} + ax^{n-2} + a^2x^{n-3} + a^3x^{n-4} +, \text{ etc.},$$

with a remainder of $2a^n$ whether n be odd or even.

In addition to the above results, observe that the number of terms in the quotient (exclusive of remainder when there is one) is always equal to the number represented by n, and that the powers of x decrease regularly by *one*, while those of a increase in the same way.

Notice also, that whether the dividend be $x^n - a^n$ or $x^n + a^n$, the terms are all plus when the divisor is $x - a$, and alternately plus and minus when the divisor is $x + a$.

By the help of these theorems we can at once write down the answers to the following :—

(1.) Divide $x^6 - a^6$ by $x - a$.

By Theorem I. $\dfrac{x^6 - a^6}{x - a} = x^5 + ax^4 + a^2x^3 + a^3x^2 + a^4x + a^5$.

(2.) Divide $x^4 + 16$ by $x + 2$.

By Theorem II. $\dfrac{x^4 + 16}{x + 2} = x^3 - 2x^2 + 4x - 8 + \dfrac{32}{x + 2}$.

(3.) Divide $x^8 - 1$ by $x^2 + 1$.

By Theorem III. $\dfrac{(x^2)^4 - 1^4}{x^2 + 1} = (x^2)^3 - (x^2)^2 + x^2 - 1$

$= x^6 - x^4 + x^2 - 1$.

EXAMPLES FOR PRACTICE.—XXIV.

(1.) Divide $x^4 - 256$ by $x - 4$.

(2.) Divide $x^5 + 32$ by $x + 2$.

(3.) Divide $81x^4 + 1$ by $3x + 1$.

(4.) Divide $64x^6 - 729$ by $2x + 3$.

(5.) Divide $125a^9 - 343b^6$ by $5a^3 + 7b^2$.

(6.) Show that $\dfrac{x^5 + a^5}{x + a} + \dfrac{x^4 - a^4}{x - a} + \dfrac{x^3 + a^3}{x + a} + \dfrac{x^2 - a^2}{x - a} + \dfrac{x + a}{x + a}$

is equal to $\dfrac{1}{a - 1}\{(a - 1)x^4 + (a^3 - 1)x^2 + (a^5 - 1)\} -$

$\dfrac{1}{a + 1}\{(a^2 - 1)x^3 + (a^4 - 1)x\}$.

82. Equations.—Equations of the character of the following will afford additional exercise in factoring and division, and are inserted mainly for that purpose:—

(1.) Given $\dfrac{x^2 - 4}{x - 2} = 7$, to find a value of x.

As $\dfrac{x^2 - 4}{x - 2}$ indicates that $x^2 - 4$ is to be divided by $x - 2$, we perform the division, and get the quotient $x + 2$, which is therefore equal to 7.

If $x + 2 = 7$, then $x = 5$.

The simplest way to divide x^2-4 by $x-2$ is to resolve it into its factors $(x+2)(x-2)$, from which we easily observe that if $x-2$ be divided out, the quotient is $x+2$.

(2.) Given $\dfrac{(x-1)^2(x^2+4)-(x^2+1)(x-2)^2}{x^2-2}=x+2$, to find a value of x.

Preparatory to dividing by x^2-2, we resolve the dividend into factors, thus—

$$\begin{aligned}&x^2(x-1)^2+4(x-1)^2-x^2(x-2)^2-(x-2)^2\\&=(x^2-x)^2-(x^2-2x)^2+(2x-2)^2-(x-2)^2\\&=(2x^2-3x)x+(3x-4)x\\&=(2x^2-3x+3x-4)x=2(x^2-2)x.\end{aligned}$$

This gives $\dfrac{2(x^2-2)x}{x^2-2}=x+2$.

Dividing, we have $2x=x+2$. $\therefore\ x=2$.

EXAMPLES FOR PRACTICE.—XXV.

In the following expressions perform the division indicated, and find the value of x which the resulting equation gives :—

(1.) $\dfrac{x^3-8}{x^2+2x+4}=1.$

(2.) $\dfrac{x^4-(x+1)^2}{x^2-(x+1)}=x^2-x+2.$

(3.) $\dfrac{x^4+x^2+1}{x^2+x+1}=x^2-3x+3.$

(4.) $\dfrac{4x^3+4}{x+1}=4x^2-5x+6.$

(5.) $\dfrac{x^3+x^2-6x}{x^2+3x}=1-2x.$

(6.) $\dfrac{x^4+64}{x^2+4x+8}=x^2-x-1.$

(7.) $\dfrac{x^4-x^2+4x-4}{x^2-x+2} = x^2-x+6.$

(8.) $\dfrac{x^5-1}{x-1} = x^4+x^3+x^2+2x.$

(9.) $\dfrac{x^4-1}{x^3-x^2+x-1} = 11.$

(10.) $\dfrac{x^3-9x^2+27x-27}{x^2-6x+9} = 0.$

(11.) $\dfrac{x^4+x^3+x+1}{x^2-x+1} = x^2+9.$

(12.) $\dfrac{(x+1)^3+(x-1)^3}{x^2+3} = 5.$

(13.) $\dfrac{16(x-1)^2-9(x+1)^2}{x-7} = 6.$

(14.) $\dfrac{x^4+x}{x^2-x+1} = \dfrac{x^3+8}{x+2}.$

(15.) $\dfrac{(x+3)^2(x^2-4)-(x-2)^2(x^2-9)}{(x+3)(x-2)} = x+9.$

(16.) $\dfrac{x^{10}+4x^8-16x^4-16x^2}{(x^2+2)^2} = x^2(x^4-4)-2(x-4).$

(17.) $\dfrac{x^5+3^5}{x+3}+3\dfrac{x^4-3^4}{x-3}-18\dfrac{x^3+3^3}{x+3}-6\dfrac{x^2-3^2}{x-3} = 78+x^4.$

(18.) $\dfrac{x^{n+2}-a^{n+2}+(x^{n-2}-a^{n-2})a^2x^2}{x^2+a^2} + \dfrac{(x-1)^4-(x-5)^4}{(x-3)^2+4}$
$= \dfrac{x^{2n}-a^{2n}}{x^n+a^n}.$

MISCELLANEOUS EXAMPLES.

Principally selected from various Examination Papers, and from other works on Algebra.

(1.) Add together $a(a+b-c)$, $b(b+c-a)$, $c(c+a-b)$.

(2.) From $3x^n+7x^{n-1}-5x^{n-2}$ take $5x^{n-1}-7x^{n-2}+3x^{n-3}$.

(3.) Find the numerical value of

$$4(ad+bc)^2-\{(a^2+d^2)-(b^2+c^2)\}^2,$$

when $a=1$, $b=2$, $c=3$, and $d=4$.

(4.) What is the value of x in the equation—

$$5x-(3+3x)=8-(-x-1)\,?$$

(5.) Remove the brackets from the following—

$$a-[5b-\{a-3(c-b)+2c-(a-2b-c)\}].$$

(6.) A workman was hired for 40 days, at 3s. 4d. per day for every day he worked, but with this condition, that for every day he did not work he was to forfeit 1s. 4d. On the whole he had £3, 3s. 4d. to receive. How many days out of the forty did he work?

(7.) Multiply $ab+cd+ac+bd$ by $ab+cd-ac-bd$.

(8.) Divide $1-ax+bx^2-cx^3+dx^4-$, etc., by $1+x$.

(9.) Add together—

$a+2x-y+24b$, $3a-4x-2y-81b$, $x+y-2a+55b$;

and subtract the result from $3a+b+3x+2y$.

(10.) Reduce the following expression to its simplest form—

$$-[+\{+(-x)\}]-\Big[-\{+[-(-x)]\}\Big].$$

(11.) Large marbles are a penny a score dearer than small ones. A boy who has bought equal numbers of both kinds finds that on the whole he has got eight for a penny. How much are the small marbles per score?

(12.) Show that $(ac\pm bd)^2+(ad\mp bc)^2=(a^2+b^2)(c^2+d^2)$,

and exemplify this identity when $a = 1 = -d$, and $b = 2 = -c$.

(13.) Multiply $x^{-2} + 3x^{-1} + 2 + x$ by $x^2 - 2x + 3$.

(14.) Resolve $3x^3 - 14x^2 - 24x$ into its simple factors.

(15.) Solve the equation—

$$(6x + 7)(x + 7) = (3x + 1)(2x + 19).$$

(16.) Two shepherds owning a flock of sheep agree to divide its value. A takes 72 sheep, whilst B takes 94 sheep and pays A £27, 10s. Find the value of a sheep.

(17.) Bracket together the coefficients of like powers of x in the following expression—

$$(mx^2 + qx + 1)^2 - (nx^2 + qx + 1)^2.$$

(18.) Add together $ax - by$, $x + y$, and $(a - 1)x - (b + 1)y$.

(19.) Find the continued product of

$mx + 2ny$, $mx - 2ny$, $mx - 3ny$, and $mx + 3ny$.

(20.) Reduce $\dfrac{x^3 - 1}{x - 1} - \dfrac{x^4 + x^2 + 1}{x^2 + x + 1}$ to its simplest form.

(21.) Simplify $(a - 2b)[9a - 2\{a + 3(a - b)\}]$.

(22.) Two persons depart at the same time from the same place to travel in the same direction round an island 36 miles in circumference; the one travels 3 miles an hour, and the other $2\frac{1}{2}$. After how many hours will they come together?

(23.) Find the sum of $(z - x)(a + b) + (z - y)(a - b)$, $(x + y)a + (x + z)b$, and $(y - z)a + (x - y)b$.

(24.) Insert the proper signs in the second side of the following identity—

$$a - b + c + d - e - f + g = a \quad \{b \quad c \quad (d \quad e \quad \overline{f \quad g})\}.$$

(25.) Simplify—

$$3(a - 2x)^2 + 2(a - 2x)(a + 2x) + (3x - a)(3x + a) - (2a - 3x)^2.$$

(26.) Resolve $m^3 - n^3 - m(m^2 - n^2) + n(m - n)^2$ into its elementary factors.

(27.) What value of x will make the difference between $(2x + 4)(3x + 4)$ and $(3x - 2)(2x - 8)$ equal to 96?

(28.) A farmer sold a certain number of bushels of wheat at 7s. 6d. per bushel, and 200 bushels of barley at 4s. 6d. per bushel, and received altogether as much as if he had sold both wheat and barley at the rate of 5s. 6d. per bushel. How much wheat did he sell?

(29.) If $a=y+z-2x$, $b=z+x-2y$, and $c=x+y-2z$, what is the value of $b^2+c^2+2bc-a^2$?

(30.) When is $a^3-b^3 = a^2+ab+b^2$?

(31.) Divide x^3-px^2+qx-r by $x-a$.

(32.) Find the coefficient of x^4 in the product of
$x^4-ax^3+a^2x^2-a^3x+a^4$ and x^2+ax+a^2.

(33.) Factor $x^8+x^6y^2+x^2y^3+y^5$.

(34.) If two numbers differ by *one*, show that the difference of their squares is equal to the sum of the numbers.

(35.) A straight lever (without weight) supports in equilibrium on a fulcrum 24 lbs. at the end of the shorter arm, and 8 lbs. at the end of the longer, but the length of the longer arm is 6 inches more than that of the shorter. Find the lengths of the arms.

Note.—The lever will be in equilibrium when the weight on one end multiplied by its distance from the fulcrum is equal to the weight on the other end multiplied by its distance from the same point.

(36.) Perform the operations indicated in the following—

$$\frac{(a+b)(a+c)-(b+d)(d+c)}{a-d}.$$

(37.) Given $a : b :: b : c$ to show that
$(a+b+c)(a-b+c) = a^2+b^2+c^2$.

(38.) If n is a positive whole number, $7^{2n+1}+1$ is divisible by 8.

(39.) Remove the brackets from the following—
$a^2-4ab[16a^2b^2-\{(a+b)^2-(a-b)^2\}^2]$.

(40.) Simplify the expressions:—

$x^{a+b+c} \times x^{a+b-c} \times x^{a-b+c} \times x^{b+c-a}$, and

$a^{m-n}b^{n-p} \times a^{n-m}b^{p-n}c$.

(41.) A weight of 6 lbs. balances a weight of 24 lbs. on a lever (supposed to be without weight) whose length is 20 inches. If 3 lbs. be added to each weight, what addition must be made to each arm of the lever so that the fulcrum may preserve its original position and the lever still retain its equilibrium?—The other conditions remaining the same, how could equilibrium be maintained by lengthening only the shorter arm?

(42.) Find the value of $x^3 - 8y^3 + 27z^3 + 18xyz$, when $x = -2y = 3z = 1$.

(43.) Expand and collect the following—

$(a+b+c)^2 + (a+b-c)^2 + (b+c-a)^2 + (c+a-b)^2$.

(44.) Divide $x^8 - xy^7 + y^8$ by $x^2 - xy + y^2$.

(45.) Show that $(4a^2 + 2ab + b^2)^2 - (4a^2 - 2ab + b^2)^2$ is equal to $8ab(4a^2 + b^2)$.

(46.) A person at a tavern borrowed as much money as he had about him, and out of the whole spent 1s. He then went to a second tavern, where he also borrowed as much as he now had about him, and out of the whole spent 1s.; and on going, in this manner, to a third and a fourth, he found that he had nothing left. How much had he at first?

(47.) Find the sum of $(a-b)x + (b-c)y + (c-a)z$, $a(y+z) + b(z+x) + c(x+y)$, and $ax + by + cz$.

(48.) Remove the brackets from the expression—

$(x-a)(x-b)(x-c) - [bc(x-a) - \{(a+b+c)x - a(b+c)\}x]$.

(49.) What is the value of x in the equation—

$(x-9)(x-7)(x-5)(x-1) = (x-2)(x-4)(x-6)(x-10)$?

(50.) Show that

$$\frac{(a+b+c)(ab+bc+ac) - abc}{a+b} = (b+c)(a+c).$$

(51.) An officer can form his men into a hollow square 4 deep, and also into a hollow square 8 deep; the front in the latter formation contains 16 men fewer than in the former. Find the number of men.

(52.) Divide

$$a^2(b+c)+b^2(a-c)+c^2(a-b)+abc \text{ by } a+b+c.$$

(53.) Multiply $a^2+2ab+b^2-c^2$ by $a^2-2ab+b^2+c^2$; and show that the result may be expressed under the form $(a^2-b^2)^2+(4ab-c^2)c^2$.

(54.) Resolve

$$(x-ab)(a-b)+(x-bc)(b-c)+(x-ca)(c-a)$$

into simple factors.

(55.) Solve the equation—

$$5x-\{8x-3(16-6x-\overline{4-5x})\} = 6.$$

(56.) Two passengers are charged for excess of luggage, 2s. 10d. and 7s. 6d. respectively. Had the luggage all belonged to one of them, he would have been charged for excess 14s. 6d. How much would each have been charged if none had been allowed free?

(57.) Show that $n(n-1)(n-2)-p(p-1)(p-2)$

$$= (n-p)\{(n+p-1)(n+p-2)-np\}.$$

(58.) Simplify the expression—

$$(a^m-a^{-n})(a^{-m}+a^n)\div(a^{m+n}-1).$$

(59.) If $a+b+c=0$, show that

$(a+b)(b+c)(c+a) = -abc$, and $a^3+b^3+c^3 = 3abc$.

(60.) Prove that $a^2b+b^2c+c^2a-ab^2-bc^2-ca^2$ is divisible without remainder by the difference between any two of the quantities a, b, c.

(61.) There was a run during a panic on two bankers, A and B. B stopped payment at the end of three days. In consequence of this the alarm increased, and the daily demand for cash on A was tripled, so that he also failed at the end of two more days. If A and B had joined their capitals, they might have stood the run as it was at first

for 7 days, at the end of which time B would have been indebted to A in £4000. What was the daily demand for cash on A's bank at the beginning of the run?

(62.) Show that

$$(1+a)^2(1+c^2)-(1+c)^2(1+a^2)=2(a-c)(1-ac).$$

(63.) Divide

$$x^3y-x^3z-xy^3-y^3z+xz^3+yz^3 \text{ by } x-y-z,$$

and resolve the quotient into factors.

(64.) What value of x satisfies the equation—

$$(8x-3)^2(x-1)=(4x-1)^2(4x-5)\text{?}$$

(65.) If a number, n, be divided into any two parts, the difference of their squares will always be equal to n times the difference of the parts. Prove this.

(66.) There are two towns, A and B, which are 131 miles distant from each other. A coach sets out from A at six o'clock in the morning, and travels at the rate of 4 miles an hour, without intermission, in the direct road towards B. At two o'clock in the afternoon of the same day a coach sets out from B to go to A, and goes at the rate of 5 miles an hour constantly. Where will they meet?

(67.) Express

$$ax^{-2}-x^{-1}+x^0(a^2-x^2)^{-1}-a^{-2}x(a^2-x^2)^{-3}+\frac{a}{a^{-1}x^{-2}}$$

with positive exponents.

(68.) Resolve the expression

$(a-b)(b+c)(c+a)+(b-c)(c+a)(a+b)+(c-a)(a+b)(b+c)$

into simple factors, and find its value when $a=1$, $b=2$, and $c=-3$.

(69.) Change $(xy-x^{-1}y^{-1})(xy^{-1}-x^{-1}y)$ into the form $(x+x^{-1})^2-(y+y^{-1})^2$, and also into $(x^2-y^2)(1-x^{-2}y^{-2})$.

(70.) In the equation

$(y+x)(y-x)=(y+2)(y+3-x)+(x-1)(x-2)(x-3)$,

what is the value of x when $y=x^2-1$?

(71.) Show that $(a+b+c)^3-(a^3+b^3+c^3)$ is equal to $3(a+b)(b+c)(c+a)$.

(72.) A and B cut packs of cards so that each took off more than he left. Now it happened that A cut off twice as many as B left, and B cut off seven times as many as A left. How were the cards cut by each?

CHAPTER V.

GREATEST COMMON MEASURE.

83. A quantity that forms a factor (Art. 36) of another is said to be a Measure of it. Thus, $5x$, being a factor of $20x^2$, is a measure of it; so also, $a-b$ is a measure of a^3-b^3.

When two or more quantities are measured by the same factor, it is called a Common Measure of them. Thus, $3xy$ is a common measure of $9x^2y$, $15xy^3$, $18x^2y^2$, and $3abxy$; so also, $m+n$ is a common measure of m^2-n^2, $m^2+2mn+n^2$, and m^3+n^3.

If two or more algebraic quantities have only one simple factor common to them all, it is, of course, their Greatest Common Measure; but if they have more than one simple factor common to them all, their Greatest Common Measure will be the product of all these factors. Thus, $9x^2y$, $15xy^3$, $18x^2y^2$, and $3abxy$ each contain the simple factors 3, x, and y; their Greatest Common Measure, therefore, is $3xy$.

Note.—3, x, y, $3x$, $3y$, xy, and $3xy$ are all *common measures* of $9x^2y$, $15xy^3$, $18x^2y^2$, and $3abxy$.

Greatest Common Measure is usually written G.C.M. for brevity.

84. When the given quantities are simple, or can readily be resolved into elementary factors, their G.C.M. may be found by inspection, and will be *the product of all their common factors, each taken once in the lowest power in which it appears in any of the quantities.*

Illustrative Examples.

(1.) Find the G.C.M. of $12abc^2$, $18a^2b^2$, and $36ab^2c$.

Here 2 and 3 are the only common factors of the coefficients, and a and b of the literal parts, therefore $6ab$ is the G.C.M. of the whole.

(2.) Find the G.C.M. of $x^3y^3 - x^2y^4$, $3x^4y^2 + 4x^3y^3$, and $4x^4y^3 - 2x^3y^4$.

Resolving into factors, we have $x^2y^3(x-y)$, $x^3y^2(3x+4y)$, and $2x^3y^3(2x-y)$. Here the only factors that appear in every one of the given expressions are powers of x and y, their lowest powers being x^2 and y^2,

$$\therefore\ x^2y^2 = \text{G.C.M.}$$

(3.) Find the G.C.M. of $a^2 - 4x^2$, $4(a^2 - 4ax + 4x^2)$, and $3a^2 - 3ax - 6x^2$.

Factoring gives us $(a+2x)(a-2x)$, $4(a-2x)^2$, and $3(a+x)(a-2x)$.

$$\text{And}\ \therefore\ a - 2x = \text{G.C.M.}$$

(4.) Find the G.C.M. of $60(x^4-1)$, $30(x^3-1)(x+1)$, $45(x-1)^2(x^3+1)$, and $75(x^2-x-2)(x^2+x-2)$.

Resolving into elementary factors, we have—

$$2^2 \times 3 \times 5 \times (x^2+1)(x+1)(x-1),$$
$$2 \times 3 \times 5 \times (x^2+x+1)(x-1)(x+1),$$
$$3^2 \times 5 \times (x-1)^2(x+1)(x^2-x+1), \text{ and}$$
$$3 \times 5^2 \times (x-2)(x+1)(x+2)(x-1).$$

Which gives the G.C.M. $= 3 \times 5 \times (x+1)(x-1)$, or $15(x^2-1)$

EXAMPLES FOR PRACTICE.—XXVI.

Find by inspection the G.C.M. of the following:—

(1.) a^3b^2 and a^2b^2c.

(2.) $12a^2c^3xy^2$ and $16ab^2x^3y^2$.

(3.) $15x^4y^3z^2$, $25x^3y^2z^4$, and $45x^2y^4z^3$.

(4.) $57m^4n^3x^2y$, $95m^3n^4y^2z$, and $38m^3n^3x^2z^2$.

(5.) $8x^3 - 4x^2y$, and $12x^2y^2 + 20x^3y$.

(6.) $2a^3bc^2 - a^2bc^2x$, and $a^3bc^3x^3 + 2a^2b^3c^3x^2 - ab^4c^3x^2$.

(7.) $2x^2 - xy$ and $4xy - 2y^2$.

(8.) $x^3 - 2x^2y$, $x^3 - 4xy^2$, and $3x^2y - 6xy^2$.

(9.) $x^4 - x^3y$, and $x^4 - 3x^3y + 2x^2y^2$.

(10.) $x^2 - 2x - 3$, $x^2 - x - 2$, and $x^2 - 1$.

(11.) $m^2x - 3mx^2 + 2x^3$, $2m^2 - 4mx + 2x^2$, and $m^3 + m^2x - 2mx^2$.

(12.) $x^6 - 2x^5 - 8x^4$, $x^5 - x^4 - 6x^3$, and $x^4 + 3x^3 + 2x^2$.

(13.) $x^2 - (y+z)^2$ and $(x+y)^2 - z^2$.

(14.) $9x^2 - (2y-z)^2$, $(3x-2y)^2 - z^2$, and $(4x-2y)^2 - (x-z)^2$.

(15.) $a^2x^3 + a^2 - x^3 - 1$ and $a^2x^2 + 2ax^2 + x^2 - a^2 - 2a - 1$.

(16.) $x^2(a-1)^2 - a^2(x-1)^2$, and $\{(x-a)+b\}^2 - \{2(a-x)-b\}^2$.

(17.) $x^{m+n}y^{2n} + x^{n+1}y^{n-1}$ and $x^{n-2}y^{m+2} - x^ny^{m+n}$.

(18.) $(a+b+c)^3 - (a^2+b^2+c^2)(a+b+c)$ and $a^2(b+c) - b^2(a+c) + c^2(a+b) + abc$.

85. There are many expressions whose factors cannot readily be obtained by inspection. In these cases their G.C.M. is found by an artifice depending on the following simple theorems.

86. I. If a quantity measures another, it will also measure any multiple of that other: that is, if P measures M, it also measures mM; for, as P is a factor of M, and the introduction of a new factor cannot strike out a former one, P must still remain a factor of mM.

II. If a quantity is a common measure of two others, it will measure both their sum and their difference.

Let P measure M and N. If $M = mP$, and $N = nP$, then $M \pm N = (m \pm n)P$, that is, P is a measure of both $M + N$ and $M - N$.

87. These two theorems are made use of in the following process :—

Suppose A and B to represent two expressions arranged according to the descending powers of some letter, and let the highest power of this letter be greater, or at least not less, in A than in B.

Divide A by B, and let the quotient be b, with the remainder C.

Now divide B by C, and let the quotient be c, with the remainder D.

Next divide C by D, and let the quotient be d, with no remainder.

Set the work down in the following order :—

$$\begin{array}{l} \mathrm{B}\,)\,\mathrm{A}\,(\,b \\ \quad\underline{b\mathrm{B}} \\ \quad\mathrm{C}\,)\,\mathrm{B}\,(\,c \\ \quad\quad\underline{c\mathrm{C}} \\ \quad\quad\mathrm{D}\,)\,\mathrm{C}\,(\,d \\ \quad\quad\quad\underline{d\mathrm{D}} \\ \quad\quad\quad 0 \end{array}$$

From this we have—

$A - bB = C$ (1), which gives $A = bB + C$ (4)
$B - cC = D$ (2), which gives $B = cC + D$ (5)
$C - dD = 0$ (3), which gives $C = dD$ (6)

From (6) we find that D is a measure of C, and being a measure of C, it is also a measure of cC (Theorem I., Art. 86), and therefore of $cC + D$ (Theorem II., Art. 86), that

is, of B (5). Being a measure of B, it is also a measure of bB, and therefore of bB + C, that is, of A (4). D, the last divisor, is therefore a common measure of A and B.

It is likewise their *greatest* common measure; for, let them have another common measure, say E, then E being a measure of B, is a measure of bB (Theorem I.); and being also a measure of A, it is a measure of A − bB (Theorem II.), that is, of C (1); and measuring C, it also measures cC, and therefore B − cC or D (2). That is, E is either equal to D, or is a factor of it, and cannot be greater.

∴ D is the G.C.M. of A and B.

Illustrative Examples.

The method of finding the G.C.M. of two quantities by this artifice can perhaps be most simply illustrated by numbers.

(1.) Required the G.C.M. of 93297 and 97681.

93297) 97681 (1
93297
4384) 93297 (21
8768
5617
4384
1233) 4384 (3
3699
685) 1233 (1
685
548) 685 (1
548
137) 548 (4
548

137, being the last divisor, is the G.C.M. of 93297 and 97681.

The work can be much shortened by being arranged in parallel columns, and the division being performed alternately forwards and backwards, thus—

21	93297	97681		1
	8768	93297		
	5617	4384		3
	4384	3699		
1	1233	685		1
	685	548		
4	548	137	=	G.C.M.
	548			

(2.) Now let it be required to find the G.C.M. of x^3-x^2-x+1 and x^3-2x-1.

$$
\begin{array}{rl}
x^3-x^2-x+1\,)\,x^3-2x-1 & (1 \\
\underline{x^3-x^2-x+1} & \\
x^2-x-2\,)\,x^3-x^2-x+1 & (x \\
\underline{x^3-x^2-2x} & \\
x+1\,)\,x^2-x-2 & (x-2 \\
\underline{x^2+x} & \\
-2x-2 & \\
\underline{-2x-2} & \\
\end{array}
$$

$\therefore\ x+1 =$ G.C.M.

By the contracted method, thus—

x	x^3-x^2-x+1 (B)	x^3-2x-1 (A)		1
	x^3-x^2-2x	x^3-x^2-x+1		
	$x+1$ (D)	x^2-x-2 (C)		$x-2$
		x^2+x		
		$-2x-2$		
		$-2x-2$		

Here (B) goes in (A) once, and leaves the remainder x^2-x-2 (C); then (C) goes in (B) x times, and leaves the remainder $x+1$ (D); and (D) being divided out of

(C), leaves no remainder. Therefore (D), that is, $x+1$, is the G.C.M. required.

(3.) Find the G.C.M. of x^4-2x^2+1 and $x^4-4x^3+6x^2-4x+1$.

$$
\begin{array}{c|l|l|c}
x^2 & x^4-2x^2+1 & x^4-4x^3+6x^2-4x+1 & 1 \\
 & x^4-2x^3+x^2 & x^4\qquad -2x^2\qquad +1 & \\
\hline
2x & 2x^3-3x^2+1 & -4x)\,-4x^3+8x^2-4x\ (C) & \\
 & 2x^3-4x^2+2x & x^2-2x+1 & \\
\hline
1 & x^2-2x+1 & & \\
 & x^2-2x+1 & & \\
\hline
\end{array}
$$

$$\therefore\ x^2-2x+1 = \text{G.C.M.}$$

In this example, the remainder (C) is observed to have the factor $4x$ common to all the terms; and as this does not form a factor in the original expressions, it cannot be a part of their G.C.M., and may be struck out before beginning the next division.

It is usual also, when the remainder is to be used as a divisor, and when the sign of its first term is minus, to change the signs of all the terms, so as to have the first one plus. (C), therefore, has been divided by $-4x$.

(4.) Find the G.C.M. of $3x^4+8x^3-4x^2-4x-3$ and $5x^4+7x^3-20x^2+11x-3$.

$$
\begin{array}{c|l|l|c}
 & 3x^4+8x^3-4x^2-4x-3\ \ (B) & 5x^4+7x^3-20x^2+11x-3\ (A) & \\
 & 19 & 3 & \\
\hline
3x & 57x^4+152x^3-\ 76x^2-76x-57 & 15x^4+21x^3-60x^2+33x-\ 9 & 5 \\
 & 57x^4+120x^3-159x^2-18x & 15x^4+40x^3-20x^2-20x-15 & \\
\hline
 & 32x^3+83x^2-58x-57\ (D) & -1)-19x^3-40x^2+53x+6\ (C) & \\
 & 19 & 19x^3+40x^2-53x-6\ (C_1) & 19x+2 \\
\hline
32 & 608x^3+1577x^2-1102x-1083 & 19x^3+38x^2-57x & \\
 & 608x^3+1280x^2-1696x-\ 192 & 2x^2+4x-6 & \\
\hline
 & 297)297x^2+594x-891\ (E) & 2x^2+4x-6 & \\
\hline
 & x^2+2x-3 & & \\
\hline
\end{array}
$$

$$\therefore\ x^2+2x-3\ =\ \text{G.C.M.}$$

Here, in dividing (A) by (B), we find that the coefficient of the first term of (A) does not exactly contain the co-

efficient of the first term of (B); and to prevent trouble from fractions, (A) is multiplied by such a number as will make its first term exactly divisible by that of (B).

The remainder (C), having its first term minus, has all its signs changed, which is equivalent to dividing it by -1. The necessity for this may be obviated by subtracting the upper line from the lower, when the subtraction of the lower from the upper would render the sign of the first term of the remainder negative. In dividing (B) by (C_1), we have again to make the first term of the dividend exactly divisible by that of the divisor, and the same thing occurs also in the next division at (D).

At (E), 297 being contained in every term is struck out.

(5.) Find the G.C.M. of $6x^4 - 24x^3 + 12x^2 + 6x$ and $2x^5 - 8x^4 + 12x^3 - 8x^2 + 2x$.

Observe that $6x$ is a factor of every term of the first expression, and $2x$ of every term of the second. These should be removed before beginning the division; but as $2x$ is a factor of both expressions, it will form a part of their G.C.M.

$$
\begin{array}{r|l|l|l}
 & x^3 - 4x^2 + 2x + 1 & x^4 - 4x^3 + 6x^2 - 4x + 1 & x \\
 & 4 & x^4 - 4x^3 + 2x^2 + x & \\
\hline
x & 4x^3 - 16x^2 + 8x + 4 & 4x^2 - 5x + 1 & 4x \\
 & 4x^3 - 5x^2 + x & 4x^2 - 4x & \\
\hline
 & -11x^2 + 7x + 4 & -x + 1 & -1 \\
 & 4 & -x + 1 & \\
\hline
-11 & -44x^2 + 28x + 16 & & \\
 & -44x^2 + 55x - 11 & & \\
\hline
 & 27)27x - 27 \ \text{(A)} & & \\
 & x - 1 & &
\end{array}
$$

$$\therefore\ 2x(x-1) = \text{G.C.M.}$$

At (A) the upper line has been subtracted from the

under, in order that the first term may be positive; also, as 27 is common to both terms of (A), it has been struck out.

EXAMPLES FOR PRACTICE.—XXVII.

Find the G.C.M. of the following:—

(1.) $x^3 - 3x^2 + 3x - 1$ and $x^3 - 2x^2 + 2x - 1$.

(2.) $x^4 - 2x^3 + 2x - 1$ and $x^4 - 2x^2 + 1$

(3.) $x^3 - 3x - 18$ and $x^4 - 9x^3 + 27x^2 - 27x$.

(4.) $2x^4 + 3x^3 - 4x^2 - 3x + 2$ and $3x^4 + 2x^3 - 6x^2 + 3x - 2$.

(5.) $6x^6 - 33x^3 - 27x$ and $8x^5 + 22x^4 + 162$.

(6.) $8ax^4 + 18ax^3 + 4ax^2 - 4ax - 8a$, and
$9x^3y + 15x^2y - 3xy + 6y$.

(7.) $2x^5 + 11x^3 + 5x$ and $2x^4 + x^3 + 10x^2 + 5x$.

(8.) $x^5 - 13x^3 - 12x^2$ and $x^5 - 16x^3 - 4x^2 + 16x$.

(9.) $72x^6 - 6x^5 + 24x^4 + 6x^3 + 12x^2$ and
$81x^5 - 54x^4 + 36x^3 + 18x^2 + 27x$.

(10.) $x^4 - (m+1)x^3 + (m+p+1)x^2 - (m+p)x + p$ and
$x^4 + (m-1)x^3 - (m+p-1)x^2 + (m+p)x - p$.

(11.) $3x^{m+2} + 2px^{m+1} - p^2x^m - 3x^2 + px$ and
$2x^{m+3} + 3px^{m+2} + p^2x^{m+1} - 2x^3 - px^2$.

(12.) $ax^{-3} - (a+b)x^{-2} + (b+c)x^{-1} - c$ and
$ax^{-4} - bx^{-3} + (a+c)x^{-2} - bx^{-1} + c$.

88. When there are three or more quantities, their G.C.M. may be obtained by first finding the G.C.M. of two of them, then of this answer and the third quantity, and so on; the last divisor being the G.C.M. of the whole.

Illustrative Example.

Find the G.C.M. of
$4a^3 - 6a^2x - 6ax^2 + 4x^3$,
$4a^3 - 2a^2x - 10ax^2 - 4x^3$, and
$4a^3 - 8a^2x - ax^2 + 2x^3$.

Remove 2 from each of the first two expressions, and observe that it will form part of their G.C.M.

$2a$	$2a^3-3a^2x-3ax^2+2x^3$	$2a^3-a^2x-5ax^2-2x^3$	1
	$2a^3-2a^2x-4ax^2$	$2a^3-3a^2x-3ax^2+2x^3$	
$-x$	$-a^2x+ax^2+2x^3$	$2x)\ 2a^2x-2ax^2-4x^3$	
	$-a^2x+ax^2+2x^3$	$a^2-ax-2x^2$	

$\therefore\ 2(a^2-ax-2x^2)$ = G.C.M. of first two.

Now find the G.C.M. of this and of third quantity. 2, not being a factor of both of these quantities, need not be written.

a	$a^2-ax-2x^2$	$4a^3-8a^2x-ax^2+2x^3$	$4a$
	a^2-2ax	$4a^3-4a^2x-8ax^2$	
x	$ax-2x^2$	$-4a^2x+7ax^2+2x^3$	$-4x$
	$ax-2x^2$	$-4a^2x+4ax^2+8x^3$	
		$3x^2)\ 3ax^2-6x^3$	
		$a-2x$	

$\therefore\ a-2x$ = the G.C.M. of the three quantities.

89. It may be observed that the G.C.M. is sometimes more easily found by reversing the order in which the expressions are stated, or at least by operating on them as if they were reversed, as in the following example :—

Find the G.C.M. of $63x^4-17x^3+17x-3$ and $49x^4+17x^2+9$.

Here, instead of making the coefficient of the first term of the one expression divisible by that of the first term of the other, we may begin at the last terms, and divide 9 by 3.

1	$63x^4-17x^3+17x-3$	$49x^4+17x^2+9$	3
	$14x^3-3x^2+x+3$	$189x^4-51x^3+51x-9$	
Adding	$3x)63x^4-3x^3-3x^2+18x$	$17x)238x^4-51x^3+17x^2+51x$	Adding
2	$21x^3-x^2-x+6$	$14x^3-3x^2+x+3$	1
	$28x^3-6x^2+2x+6$	$7x^2-5x+3$	
Subtr.	$x)\ 7x^3-5x^2+3x$	$14x^3-10x^2+6x$	$2x$, Sub.
	$7x^2-5x+3$	$14x^3-10x^2+6x$	

$\therefore\ 7x^2-5x+3$ = G.C.M.

EXAMPLES FOR PRACTICE.—XXVIII.

Find the G.C.M. of the following :—

(1.) $6x^4 - 3x^3 - 20x^2 + x + 6$, $10x^3 - 3x^2 - 31x - 6$, and $15x^4 - 27x^3 - 11x^2 + 9x + 2$.

(2.) $35x^5 + x^4 - 41x^3 - x^2 + 6x$, $70x^5 - 33x^4 - 118x^3 + 3x^2 + 18x$, and $30x^4 - 57x^3 - 12x^2 + 57x - 18$.

(3.) $x^5 - 4x^4 + 4x^3 + 2x^2 - 5x + 2$, $x^6 + 3x^5 - 4x^4 - 10x^3 + 9x^2 + 7x - 6$, $x^6 + x^5 - 2x^4 - x^2 - x + 2$, and $x^5 + 2x^4 - 2x^2 - x$.

CHAPTER VI.

FRACTIONS.

0. We have seen (Art. 62) that one quantity standing r another with a line between them indicates the divi- ı of the upper line by the under. If, however, the ıer will not divide by the under, the expression is ed a fraction, as $\frac{a}{b}$, $\frac{x+1}{x^2+1}$.

Even in cases where the division can be performed the ression may be spoken of as a fraction, since it has form of one, and can be treated as one. Thus, $\frac{x^3+8}{x+2}$ may be spoken of and treated as fractions. ı integral forms, a, $x+y$, etc., may be made to assume fractional form by placing them over the figure 1— s, $\frac{a}{1}$, $\frac{x+y}{1}$.

The expression $\frac{a}{b}$ indicates that the number or quantity s to be divided into b parts: thus, if a represent 3 ds, and b the number 4, $\frac{a}{b}$ means that 3 yards are to divided into 4 pieces. It may also mean that one yard

is to be divided into four pieces, and that three of these are to be taken.

$\frac{a}{b}$ may thus signify either the $\frac{1}{b}$th part of a units, or a times the $\frac{1}{b}$th part of one unit.

From this latter view of a fraction, the upper line is named the numerator, and the lower the denominator.

91. Reduction of Fractions.—*Theorem:* A fraction is not altered in value by having both its numerator and denominator multiplied by the same number.

For, let $\frac{a}{b}$ be a fraction and m any multiplier, if the unit be divided into mb instead of b parts, each part will be m times smaller than before, and it will take m times as many of them to make up a fraction equal to the original one; as there were a of them at first, there must now be ma of them,—that is,

$$\frac{a}{b} = \frac{ma}{mb}. \quad \text{So, } \frac{x}{1} = \frac{4x}{4} = \frac{x^3}{x^2} = \frac{x^4 - x}{x^3 - 1}, \text{ etc.}$$

Theorem: A fraction is not altered in value by having both its numerator and denominator divided by the same quantity.

For, if $\frac{a}{b} = \frac{ma}{mb}$, plainly $\frac{ma}{mb} = \frac{a}{b}$.

$$\text{So, } \frac{9ab^2c^3}{12a^3b^2c} = \frac{3c^2}{4a^2}, \text{ and } \frac{x^4 - 2x^2 + 1}{x^6 - 1} = \frac{x^2 - 1}{x^4 + x^2 + 1}.$$

92. As a deduction from these two theorems, the signs of all the terms of a fraction may be changed without altering its value, for this is equivalent to either multi-

plying or dividing both the numerator and denominator by -1.

$$\frac{a}{b} = \frac{-a}{-b}, \quad \frac{-x}{y} = \frac{x}{-y}, \quad \frac{a-c}{x-y} = \frac{c-a}{y-x}, \text{ and } \frac{a-x}{x-a} = -\frac{a-x}{a-x}.$$

93. To reduce an integral quantity to the form of a fraction having any required denominator: Place the given quantity upon 1, and multiply the upper and under lines by the new denominator.

Illustrative Example.

Reduce ab and a^2-b^2 to the form of fractions having $2ab$ for a denominator.

$$\frac{ab}{1} \times \frac{2ab}{2ab} = \frac{2a^2b^2}{2ab}.$$

$$\frac{a^2-b^2}{1} \times \frac{2ab}{2ab} = \frac{2a^3b-2ab^3}{2ab}.$$

EXAMPLES FOR PRACTICE.—XXIX.

(1.) Express xy and x^2+y^2 as fractions having the denominator mn.

(2.) Reduce $a^2-2ax+x^2$ to the form of a fraction with the denominator $(a+x)^2$.

(3.) Represent x^3+x^2+x+1 as a fraction whose denominator is $x-1$.

(4.) What fraction having the denominator $x+y+z$ is equal to $x(x-y)+y(y-z)+z(z-x)$?

(5.) Form $x^m-x^{m-n}y^m$ into a fraction having the denominator x^n+y^m.

(6.) Write the fraction $\frac{x^3-2x^2+2x-1}{a-b}$ with the denominator $b-a$.

94. A fraction is said to be in its simplest or lowest

terms when its numerator and denominator have no common factor.

To reduce a fraction to its lowest terms, strike out every factor common to both numerator and denominator, or divide each of them by their G.C.M.

Illustrative Examples.

(1.) Reduce $\frac{a^3b^2c}{ab^2c^3}$ to its lowest terms.

Here the common factors are a, b^2, and c; these being struck out of upper and under lines, the fraction in its lowest terms becomes $\frac{a^2}{c^2}$.

(2.) Express $\frac{x^4y^2-x^4-y^2+1}{x^2y^4-x^2-y^4+1}$ in its simplest terms.

Resolving each line into factors, we have

$$\frac{x^4(y^2-1)-(y^2-1)}{x^2(y^4-1)-(y^4-1)}=\frac{(x^4-1)(y^2-1)}{(x^2-1)(y^4-1)}=\frac{(x^2+1)(x^2-1)(y^2-1)}{(x^2-1)(y^2+1)(y^2-1)}$$
$$=\frac{x^2+1}{y^2+1},$$

by cancelling out the common factors, x^2-1 and y^2-1.

(3.) Simplify $\frac{2x^3-7x^2+9}{x^4-2x^3-6x-9}$.

First find G.C.M.

$$\begin{array}{c|l|l}
2 & 2x^3-7x^2+9 & x^4-2x^3-6x-9 \\
 & 2x^3-14x-12 & 2 \\
\hline
 & 7)\ 7x^2-14x-21 & 2x^4-4x^3-12x-18 \quad | \quad x \\
 & x^2-2x-3 & 2x^4-7x^3+9x \\
\hline
 & & 3)\ 3x^3-21x-18 \\
 & & x^3-7x-6 \quad | \quad x+2 \\
 & & x^3-2x^2-3x \\
\hline
 & & 2x^2-4x-6 \\
 & & 2x^2-4x-6
\end{array}$$

$\therefore\ x^2-2x-3=$ G.C.M.

Now divide numerator and denominator by x^2-2x-3.

$$x^2-2x-3 \;)\; 2x^3-7x^2+9 \;(\; 2x-3$$
$$\underline{2x^3-4x^2-6x}$$
$$-3x^2+6x+9$$
$$-3x^2+6x+9$$

$$x^2-2x-3 \;)\; x^4-2x^3-6x-9 \;(\; x^2+3$$
$$\underline{x^4-2x^3-3x^2}$$
$$3x^2-6x-9$$
$$3x^2-6x-9$$

$$\therefore \frac{2x^3-7x^2+9}{x^4-2x^3-6x-9} = \frac{2x-3}{x^2+3}.$$

(4.) Reduce $\frac{a-c}{(b-c)(c-a)}$, and $-\frac{(a-b)(x-a)(x-b)}{(a-x)(b-x)(c-x)}$, to their simplest forms.

The first, by re-arrangement of numerator, becomes $\frac{-(c-a)}{(b-c)(c-a)}$, and by cancelling out the common factor $c-a$, it changes into $\frac{-1}{b-c}$, in which form it may be left; or it may be still further changed into $\frac{1}{c-b}$ by multiplying (or dividing) both upper and under lines by -1 (Art. 92).

The process of simplifying this expression might have been begun by reversing the order and signs of all the terms of the denominator, thus, $\frac{a-c}{(c-b)(a-c)}$, and cancelling out $a-c$, we have $\frac{1}{c-b}$ as before.

95. Observe that if only one of the factors of the denominator had been changed, as, say, $b-c$ into $c-b$, the sign of the whole expression would have become negative; for, if the sign of one of two factors be changed,

the sign of their product is changed, and the denominator being thus made negative, the numerator must be made negative too. The fraction would thus become $\frac{-(a-c)}{(c-b)(c-a)}$ or $-\frac{a-c}{(c-b)(c-a)}$. If now the other factor $c-a$ be written $a-c$, the sign of the denominator will be again changed, and to preserve the sign of the fraction unaltered, the numerator must become positive, so that we have $\frac{a-c}{(b-c)(c-a)} = \frac{a-c}{(c-b)(a-c)}$.

From this we have the rule that if the signs of any odd number of factors in numerator and denominator together be changed (as, say, three in numerator, or two in numerator and one in denominator), the sign of the fraction is changed; but if the signs of an even number be changed (as, say, one in numerator and three in denominator), the sign of the fraction remains unaltered.

The second fraction $-\frac{(a-b)(x-a)(x-b)}{(a-x)(b-x)(c-x)}$ becomes $\frac{(b-a)(a-x)(b-x)}{(a-x)(b-x)(c-x)} = \frac{b-a}{c-x}$, or $\frac{(a-b)(x-a)(x-b)}{(x-a)(x-b)(x-c)} = \frac{a-b}{x-c}$.

Here, as the signs of three factors have been changed, the sign of the fraction has changed.

EXAMPLES FOR PRACTICE.—XXX.

Reduce the following fractions to their simplest terms:—

(1.) $\frac{12ax^2y^3}{16a^2cxy^2}$.

(2.) $\frac{361m^3x^4yz}{380m^4x^5}$.

(3.) $\frac{7axy}{3x^2y-5xy^2}$

(4.) $\frac{4x-8x^2+12x^3}{4x^3}$.

(5.) $\frac{3x^2y-6xy^2}{6x^2y+3xy^2}$.

(6.) $\frac{35x^3y+28x^2y^2}{49x^2y^2-42x^2yz}$.

(7.) $\dfrac{x^2-7x+12}{x^2-8x+15}$.

(8.) $\dfrac{y^2-7yz+6z^2}{y^2-5yz-6z^2}$.

(9.) $\dfrac{a^3-ab^2}{a^2b-b^3}$.

(10.) $\dfrac{a^3x-ax^3}{a^3x+a^2x^2-2ax^3}$.

(11.) $\dfrac{a^3-x^3}{a^3+a^2x+ax^2}$.

(12.) $\dfrac{2m^4y^2-4m^2y+8}{m^6y^3+8}$.

(13.) $\dfrac{a^4-x^2y^2}{a^6-x^3y^3}$.

(14.) $\dfrac{(2x)^2-(y+2z)^2}{(2x+y)^2-(2z)^2}$.

(15.) $\dfrac{x^2+(1-y)xy-y^3}{x^2-y^4}$.

(16.) $\dfrac{mp+mq-np-nq}{mr-nr+ms-ns}$.

(17.) $\dfrac{ax+2by-2bx-ay}{ax-2by+ay-2bx}$.

(18.) $\dfrac{6x^2-13x+6}{6x^2+5x-6}$.

(19.) $\dfrac{x^3-2x^2-4x+8}{x^4-4x^3+8x^2-16x+16}$.

(20.) $\dfrac{x^3-2x^2-x+2}{x^3-3x-2}$.

(21.) $\dfrac{10x^3-7x^2-2x+8}{15x^3+2x^2+7x+12}$.

(22.) $\dfrac{a^2+a(b-1)x+(ac-b)x^2-cx^3}{ab+(b^2-a)x+b(c-1)x^2-cx^3}$.

(23.) $\dfrac{x^{p+1}-2x^p-3x^{p-1}-x+3}{x^{p+2}+3x^{p+1}+2x^p-x^2-2x}$.

(24.) $\dfrac{mx^2-(m-n)x+(m-n)-(m-n)x^{-1}-nx^{-2}}{px^2-(p+q)x+(p+q)-(p+q)x^{-1}+qx^{-2}}$.

Multiplication and Division.

96.—I. Multiplication and Division of a Fraction by an Integral Quantity.

Let $\frac{a}{b}$ be any fraction; m any integral quantity. As b merely indicates the kind of parts into which unity has been divided, while a shows the number of these that are to be dealt with (Art. 90), it is evident that the fraction

will be multiplied or divided according as a is multiplied or divided.

Therefore $\frac{a}{b}$ multiplied by m becomes $\frac{a \times m}{b}$ or $\frac{am}{b}$, and $\frac{a}{b}$ divided by m becomes $\frac{a \div m}{b}$.

If a will not divide exactly by m, the fraction $\frac{a}{b}$ may be written $\frac{am}{bm}$ (Art. 91); and the numerator being now divided by m, we get the quotient $\frac{a}{bm}$.

From this we have the following

RULES.—To multiply a fraction by an integral quantity, multiply the numerator by it; and to divide a fraction by an integral quantity, multiply the denominator by it.

Illustrative Examples.

(1.) Multiply $\frac{3ax}{4by}$ by $2xy$.

$$\frac{3ax}{4by} \times 2xy = \frac{6ax^2y}{4by} = \frac{3ax^2}{2b} \text{ (Art. 94).}$$

(2.) Multiply $\frac{x^2+a^2}{x^3+a^3}$ by $x^2 - ax + a^2$.

$$\frac{x^2+a^2}{x^3+a^3} \times (x^2 - ax + a^2) = \frac{(x^2+a^2)(x^2 - ax + a^2)}{x^3+a^3}$$

$$= \frac{(x^2+a^2)(x^2 - ax + a^2)}{(x^2 - ax + a^2)(x+a)} = \frac{x^2+a^2}{x+a}.$$

(3.) Divide $\frac{6a^2x^2}{5mn}$ by $3axy$.

$$\frac{6a^2x^2}{5mn} \div 3axy = \frac{6a^2x^2}{5mn \times 3axy} = \frac{2ax}{5mny}.$$

(4.) Divide $\frac{a^2-(b+c)^2}{a-b+c}$ by $a+b+c$.

$$\frac{a^2-(b+c)^2}{a-b+c} \div (a+b+c) = \frac{a^2-(b+c)^2}{(a-b+c)(a+b+c)}$$

$$= \frac{(a+b+c)(a-b-c)}{(a-b+c)(a+b+c)} = \frac{a-b-c}{a-b+c}.$$

(5.) Find the $\frac{1}{rn}$th part of tm times $\frac{r(a+b)}{t(a-b)}$.

tm times $\frac{r(a+b)}{t(a-b)} = \frac{tmr(a+b)}{t(a-b)} = \frac{mr(a+b)}{a-b}$: of which the $\frac{1}{rn}$th part is $\frac{mr(a+b)}{rn(a-b)}$ or $\frac{m(a+b)}{n(a-b)}$.

97.—II. Multiplication of a Fraction by a Fraction.

Let it be required to multiply $\frac{a}{b}$ by $\frac{m}{n}$.

If we multiply $\frac{a}{b}$ by m, we shall have $\frac{am}{b}$ (Art. 96); but as $\frac{m}{n}$ is the $\frac{1}{n}$th part of m, the required product must be the $\frac{1}{n}$th part of $\frac{am}{b}$, that is, $\frac{am}{bn}$ (Art. 96).

If this fraction were to be still further multiplied by $\frac{x}{y}$, we should in the same manner have the $\frac{1}{y}$th part of $\frac{amx}{bn}$ or $\frac{amx}{bny}$; and so on for any number of fractions.

From this we obtain the following

RULE FOR MULTIPLYING FRACTIONS TOGETHER.—Take the product of the numerators for a new numerator, and the product of the denominators for a new denominator.

Illustrative Examples.

(1.) Multiply $\frac{8ax^2}{9y^3z^2}$ by $\frac{3by^2z^2}{4a^2x}$.

$$\frac{8ax^2}{9y^3z^2} \times \frac{3by^2z^2}{4a^2x} = \frac{24abx^2y^2z^2}{36a^2xy^3z^2} = \frac{2bx}{3ay}.$$

Here, the numerators having been multiplied together, and the denominators together, the resulting fraction has been reduced to its lowest terms, by Art. 94; but this can be done before the multiplication is actually performed, by striking out the factors common to the numerator of the one and the denominator of the other—thus:

$$\frac{8ax^2}{9y^3z^2} \times \frac{3by^2z^2}{4a^2x} = \frac{2x}{3y} \times \frac{b}{a} = \frac{2bx}{3ay}.$$

(2.) Multiply $\frac{x^2-7x+12}{x^3-8}$ by $\frac{x^2+2x+4}{x-4}$.

Resolve into factors whatever terms are capable of resolution, and strike out those common to upper and under lines.

$$\frac{x^2-7x+12}{x^3-8} \times \frac{x^2+2x+4}{x-4} = \frac{(x-3)(x-4)}{(x^2+2x+4)(x-2)} \times \frac{x^2+2x+4}{x-4}$$
$$= \frac{x-3}{x-2} \times \frac{1}{1} = \frac{x-3}{x-2}.$$

(3.) Find the continued product of—

$\frac{x^2-2x+2}{x^2+2x-3}$, $\frac{x^2-9}{x^4+4}$, $\frac{x^2+2x+2}{x^2-2x-3}$, and $(x-1)^2$.

$$\frac{x^2-2x+2}{(x+3)(x-1)} \times \frac{(x+3)(x-3)}{(x^2-2x+2)(x^2+2x+2)} \times \frac{x^2+2x+2}{(x-3)(x+1)}$$
$$\times \frac{(x-1)^2}{1} = \frac{x-1}{x+1}.$$

98.—III. Division of a Fraction by a Fraction.

Let it be required to divide $\frac{a}{b}$ by $\frac{m}{n}$. First divide by m; this gives $\frac{a}{bm}$ (Art. 96); but as we have divided by a quantity n times too great, our quotient must be n times too small. We therefore require to multiply this quotient by n; thus we obtain $\frac{an}{bm}$ or $\frac{a}{b} \times \frac{n}{m}$.

Here we observe that the divisor *inverted* has become a multiplier to the quantity to be divided; and from this we derive the following

RULE.—To divide by a fraction, invert the divisor, and proceed as in multiplication.

Illustrative Examples.

(1.) Divide $\frac{4a^3xy^2}{15bc}$ by $\frac{6a^4xy}{5c^2z}$.

$$\frac{4a^3xy^2}{15bc} \div \frac{6a^4xy}{5c^2z} = \frac{4a^3xy^2}{15bc} \times \frac{5c^2z}{6a^4xy}$$

$$= \frac{2y}{3b} \times \frac{cz}{3a} = \frac{2cyz}{9ab}.$$

(2.) Divide $\frac{a^2-b^2-c^2+2bc}{a^2+ac-b^2-bc}$ by $\frac{a^2-2ac+c^2-b^2}{a^2+ab-bc-c^2}$.

$$\frac{a^2-(b^2+c^2-2bc)}{a^2+ab+ac-ab-b^2-bc} \div \frac{(a^2-2ac+c^2)-b^2}{a^2+ab+ac-ac-bc-c^2}$$

$$= \frac{a^2-(b-c)^2}{a(a+b+c)-b(a+b+c)} \times \frac{a(a+b+c)-c(a+b+c)}{(a-c)^2-b^2}$$

$$= \frac{(a+b-c)(a-b+c)}{(a-b)(a+b+c)} \times \frac{(a-c)(a+b+c)}{(a+b-c)(a-b-c)}$$

$$= \frac{(a-b+c)(a-c)}{(a-b)(a-b-c)}.$$

(3.) Divide $\dfrac{(x-y)^3-x^3+y^3}{(x+y)^5-x^5-y^5}$ by $\dfrac{(x-y)^5-x^5+y^5}{(x+y)^7-x^7-y^7}$.

Resolve the dividend into factors—

$$\frac{(x-y)^3-(x^3-y^3)}{(x+y)^5-(x^5+y^5)}$$

$$= \frac{(x-y)\{(x-y)^2-(x^2+xy+y^2)\}}{(x+y)\{(x+y)^4-(x^4-x^3y+x^2y^2-xy^3+y^4)\}}$$

$$= \frac{(x-y)(x^2-2xy+y^2-x^2-xy-y^2)}{(x+y)(x^4+4x^3y+6x^2y^2+4xy^3+y^4-x^4+x^3y-x^2y^2+xy^3-y^4)}$$

$$= \frac{(x-y)(-3xy)}{(x+y)(5x^3y+5x^2y^2+5xy^3)} = \frac{-3xy(x-y)}{5xy(x+y)(x^2+xy+y^2)}.$$

Resolve the divisor into factors—

$$\frac{(x-y)^5-(x^5-y^5)}{(x+y)^7-(x^7+y^7)}$$

$$= \frac{(x-y)\{(x-y)^4-(x^4+x^3y+x^2y^2+xy^3+y^4)\}}{(x+y)\{(x+y)^6-(x^6-x^5y+x^4y^2-x^3y^3+x^2y^4-xy^5+y^6)\}}$$

$$= \frac{(x-y)(-5x^3y+5x^2y^2-5xy^3)}{(x+y)(7x^5y+14x^4y^2+21x^3y^3+14x^2y^4+7xy^5)}$$

$$= \frac{-5xy(x-y)(x^2-xy+y^2)}{7xy(x+y)(x^4+2x^3y+3x^2y^2+2xy^3+y^4)}$$

$$= \frac{-5xy(x-y)(x^2-xy+y^2)}{7xy(x+y)(x^2+xy+y^2)^2}.$$

Then $\dfrac{(x-y)^3-x^3+y^3}{(x+y)^5-x^5-y^5} \div \dfrac{(x-y)^5-x^5+y^5}{(x+y)^7-x^7-y^7}$

$$= \frac{-3xy(x-y)}{5xy(x+y)(x^2+xy+y^2)} \div \frac{-5xy(x-y)(x^2-xy+y^2)}{7xy(x+y)(x^2+xy+y^2)^2}$$

$$= \frac{-3xy(x-y)}{5xy(x+y)(x^2+xy+y^2)} \times \frac{7xy(x+y)(x^2+xy+y^2)^2}{-5xy(x-y)(x^2-xy+y^2)}$$

$$= \frac{21}{25} \cdot \frac{x^2+xy+y^2}{x^2-xy+y^2}.$$

EXAMPLES FOR PRACTICE.—XXXI.

(1.) Multiply $\frac{5ac^2x^2}{6b^2yz}$ by $3aby^3$.

(2.) Multiply $\frac{24}{x^3-4x}$ by x^2-4x+4.

(3.) Multiply $\frac{a^3-a^2x}{b^3-b^2x}$ by $\frac{b^4}{a^4}$.

(4.) Multiply $\frac{x^4-3x^3+2x^2}{x^2+1}$ by $\frac{2x^2+2}{x^3-x^2-2x}$.

(5.) Multiply $\frac{x^3-2x^2+x}{x^3+a^3}$ by $\frac{x^2-ax+a^2}{x^2-x}$.

(6.) Multiply $\frac{x^2+xy}{3x-3y}$ by $\frac{2x-2y}{xy+y^2}$.

Find the simplest value of the following:—

(7.) $\frac{2x^2-3x+4}{x^2-4x-5} \times \frac{2x^2-4x+6}{x^2-9} \times \frac{x^2-2x-15}{4x^2-6x+8}$.

(8.) $\frac{x^2-x+1}{x^2-4x} \times \frac{x^2+x-20}{x^4+x^2+1} \times \frac{x^2+x+1}{x^2+5x}$.

(9.) $\frac{x^6-y^3z^3}{a^3+b^6} \times \frac{a^2-ab^2+b^4}{x^2-yz} \times \frac{a+b^2}{x^4+x^2yz+y^2z^2}$.

(10.) $\frac{2ac-ad-6bc+3bd}{ab-(a+b)x+x^2} \times \frac{ac-(a+c)x+x^2}{ac-3bc+2ad-6bd} \times \frac{bc+2cd+2bd+c^2}{2c^2+dx-2cx-cd}$.

(11.) $\frac{(a-b)(c-2b)}{(2a-b)(c-b)} \times \frac{(a-2c)(b-c)}{(2b-c)(a-c)} \times \frac{(c-a)(b-2a)}{(b-a)(2b-a)}$.

(12.) $\frac{(x-y)^2-(z-1)^2}{(a-b)^2-(c-1)^2} \times \frac{(a-b+c)^2-1}{1-(x-y-z)^2}$.

(13.) Divide $\frac{6}{7}x^2y^2z$ by 3.

(14.) Divide $\frac{x^4y^2z^4}{a^2b^2c^2}$ by $5a^2y^2z^2$.

(15.) Divide $\frac{mnx^2}{ab^2c}$ by $\frac{pqy^2}{a^2bc^2}$.

(16.) Divide $\frac{a^3-a^2b}{ab^2+b^3}$ by $\frac{a^2}{b^2}$.

(17.) Divide $\frac{x^3-8}{x^3+x}$ by $\frac{x^2+2x+4}{x^2+1}$.

(18.) Divide $\frac{x^4-4x^3+6x^2-4x+1}{x^3+3x^2+3x+1}$ by $\left(\frac{x-1}{x+1}\right)^4$.

(19.) Simplify $\frac{x^2-x-6}{x^2+4x+3} \times \frac{x^2-1}{x^2-4} \div \frac{x^2-5x+4}{x^2+x-6}$.

(20.) Simplify $\frac{x^2+x+1}{x^2+1} \times \frac{x^2-x+1}{x^2-1} \div \frac{4x}{(x^2+x+1)^2-(x^2-x+1)^2}$.

(21.) Simplify $\frac{(1+x+x^2)^2-(1-x-x^2)^2}{(1+x+x^2+x^3)^2} \div \frac{(1+x)^2-(1-x)^2}{(1+x)^2+(1-x)^2}$.

(22.) Simplify $\frac{(x^2-a^2)(y+b)^2-(y^2-b^2)(x+a)^2}{(a+b+x+y)^2-(a-b+x-y)^2} \div \frac{bx-ay}{2}$.

(23.) Simplify $\frac{(x-y)^5-x^5+y^5}{x^6+x^3y^3+x^3+y^3} \div \frac{(x-y)^3-x^3+y^3}{x^4+x^3y+x+y}$.

(24.) Simplify $\frac{x^n}{y^{n-3}} \cdot \left(\frac{x^2-y^2}{x^2+y^2}\right)^{n-1} \times (x^5y-xy^5)^n \div \frac{x^2y^2-y^4}{x^4+x^2y^2}$

CHAPTER VII.

LEAST COMMON MULTIPLE.

99. When one quantity is exactly divisible by another, it is said to be a multiple of that other.

When one quantity is exactly divisible by each of two or more quantities, it is called a common multiple of these quantities. Thus, $12a^3b^2x$ is a multiple of $6a^2b^2$, $4abx$, and $3a^3x$, and is therefore called a common multiple of them.

Observe that the number of common multiples of any set of quantities is unlimited. In the above example, *any* multiple of $12a^3b^2x$ must also be a multiple of $6a^2b^2$, $4abx$, and $3a^3x$; and as any number of multiples of $12a^3b^2x$ may be found, there will be any number of common multiples of $6a^2b^2$, $4abx$, and $3a^3x$.

Almost invariably when a common multiple of two or more quantities is to be found, it is the least one that is required.

The expression Least Common Multiple is usually contracted to L.C.M.

Let us suppose that the L.C.M. of $12ab^2c$, $15a^3x$, $18b^2c^2$, and $20c^3x$ is wanted.

As the L.C.M. of these quantities is to be divisible by

12, 15, 18, and 20, it must contain every factor that enters into the composition of any one of these coefficients.

Now the factors of 12 are 3 and 2^2; of 15, 3 and 5; of 18, 2 and 3^2; and of 20, 5 and 2^2.

Plainly the smallest number that contains each of these factors is $2^2 \times 3^2 \times 5$, or 180.

Similarly the L.C.M. of the literal parts must contain a^3, b^2, c^3, and x; for, as the L.C.M. is to be divisible by each of the given quantities, it must contain in its highest power every factor that appears in any one of them.

$180a^3b^2c^3x$ is therefore the L.C.M. of $12ab^2c$, $15a^3x$, $18b^2c^2$, and $20c^3x$.

From this we deduce the following

RULE.—To find the L.C.M. of any number of quantities, take every elementary factor that appears in any of the quantities once in its highest power; the product of these will be the L.C.M. required.

Illustrative Examples.

(1.) Find the L.C.M. of $16a^6x^2$, $18b^4x^2y^2$, $24c^2x^3z^3$, and $40x^3y^3z^2$.

The coefficients being resolved into factors give 2^4, 2×3^2, $2^3 \times 3$, and $2^3 \times 5$; and taking the highest powers of 2, 3, and 5 that appear in any of them, as well as the highest powers of the literal factors, we have the required

$$\begin{aligned}\text{L.C.M.} &= 2^4 \times 3^2 \times 5 \times a^6b^4c^2x^3y^3z^3 \\ &= 720a^6b^4c^2x^3y^3z^3.\end{aligned}$$

(2.) Find the L.C.M. of x^3+x^2-4x-4, $x^3+2x^2-9x-18$, and x^3-x^2-5x-3.

Resolving each into factors gives—

$$\begin{aligned}x^3+x^2-4x-4 &= x^2(x+1)-4(x+1)=(x+1)(x^2-4) \\ &= (x+1)(x+2)(x-2). \\ x^3+2x^2-9x-18 &= x^2(x+2)-9(x+2)=(x+2)(x^2-9) \\ &= (x+2)(x+3)(x-3).\end{aligned}$$

$$\begin{aligned} x^3 - x^2 - 5x - 3 &= x^3 - 3x^2 + 2x^2 - 6x + x - 3 \\ &= x^2(x-3) + 2x(x-3) + (x-3) \\ &= (x^2 + 2x + 1)(x-3) = (x+1)^2(x-3). \end{aligned}$$

Taking each factor once in its highest power, we have—

$$(x+1)^2(x+2)(x-2)(x+3)(x-3) = (x+1)^2(x^2-4)(x^2-9)$$

$$= \text{L.C.M.}$$

(3.) Find the L.C.M. of $8 - x^3$, $16 + 4x^2 + x^4$, and $8 + x^3$.

$$\begin{aligned} 8 - x^3 &= (2-x)(4+2x+x^2) \\ 16 + 4x^2 + x^4 &= 16 + 8x^2 + x^4 - 4x^2 = (4+x^2)^2 - (2x)^2 \\ &= (4+2x+x^2)(4-2x+x^2) \\ 8 + x^3 &= (2+x)(4-2x+x^2) \\ \therefore \text{L.C.M.} &= (2-x)(4+2x+x^2)(2+x)(4-2x+x^2) \\ &= (8-x^3)(8+x^3) = 64 - x^6. \end{aligned}$$

EXAMPLES FOR PRACTICE.—XXXII.

Find the L.C.M. of the following :—

(1.) $8a^3m^2x$, $12a^2m^3y$, and $15amz$.

(2.) ab^2x^2y, bc^2y^2z, cdz^2x, $dexy^3z$, and efx^2z^2.

(3.) m^2x^2, $30x^3y$, $36y^2z$, xyz, and 45.

(4.) $b(b+c)$, $c(b+c)$, and abc.

(5.) $42(x+y)$, $35(x-y)$, and $28(x^2-y^2)$.

(6.) $x^2(x-1)$, $x(x^2-1)$, and x^3-1.

(7.) x^2+x-6, x^2-5x+6, and x^2-x-6.

(8.) $4x^2-1$, $1-2x$, $1-x$, and $1+x$.

(9.) $x^2-4x-21$, x^3+x^2-56x, and $x^4+11x^3+24x^2$.

(10.) x^4-x^3+x-1, x^4+x^2+1, and x^4+x^3-x-1.

(11.) $(x^3+x^2+x)^2-(x^2+x+1)^2$, and $x^5+x^3-x^2-1$.

(12.) $b^2-(a+c)^2$, $(c-b)^2-a^2$, $c^2-(a+b)^2$, $x-y$, and y^2-x^2.

100. When the given quantities cannot be readily resolved into factors, their L.C.M. may be obtained in the following manner :—

Find the G.C.M. of two of them; the product of these two divided by their G.C.M will give their L.C.M. Next find the G.C.M. of this answer and the third quantity; the product of these two divided by *their* G.C.M. will give the L.C.M. of the three quantities. In the same way it may be obtained for any number of quantities.

That this method yields the required result will appear from the following investigation :—

Let A and B be two quantities having M for their G.C.M. and R for their L.C.M.; then we may put $A = aM$, and $B = bM$.

Now every factor of either A or B will be contained in abM, and therefore abM is a common multiple of A and B; and as a and b (from the nature of a G.C.M.) have no common measure but unity, abM is the least quantity that will exactly contain aM and bM, and is therefore the L.C.M. of A and B.

$$\text{Since} \quad A \times B = aM \times bM = abM^2,$$

$$\therefore \quad \frac{A \times B}{M} = abM = \text{L.C.M. of A and B} = R;$$

that is (as above), the product of two quantities divided by their G.C.M. gives their L.C.M.

If there be a third quantity, C, the L.C.M. of R and C, which we may call S, will be $\frac{R \times C}{N}$, where N is the G.C.M. of R and C. As any multiple of R is also a multiple of A and B (Art. 99), S must be a common multiple of A, B, and C; and it is also their *least* common multiple, for it is the L.C.M. of R and C, and R is the L.C.M. of A and B.

Illustrative Example.

What is the L.C.M. of $6x^3 - x^2 - 5x + 2$, $6x^4 + 5x^3 - 3x^2 - 2x$, and $4x^3 - 4x^2 - x + 1$?

The G.C.M. of the first two is found to be $3x^2+x-2$; therefore their

$$\text{L.C.M.} = \frac{(6x^3-x^2-5x+2)(6x^4+5x^3-3x^2-2x)}{3x^2+x-2} \quad \text{(A)}$$
$$= (2x-1)(6x^4+5x^3-3x^2-2x)$$
$$= 12x^5+4x^4-11x^3-x^2+2x.$$

(A) The denominator $3x^2+x-2$ being a measure of each of the two factors of the numerator, is divided out of one of them before the multiplication indicated is performed.

The G.C.M. of $12x^5+4x^4-11x^3-x^2+2x$ and $4x^3-4x^2-x+1$ is next found to be $4x^2-1$, and therefore the L.C.M. of the three quantities

$$= \frac{(12x^5+4x^4-11x^3-x^2+2x)(4x^3-4x^2-x+1)}{4x^2-1}$$
$$= (12x^5+4x^4-11x^3-x^2+2x)(x-1)$$
$$= 12x^6-8x^5-15x^4+10x^3+3x^2-2x.$$

EXAMPLES FOR PRACTICE.—XXXIII.

(1.) Find the L.C.M. of x^3-3x+2 and x^3+x^2-5x+3.

(2.) Obtain the L.C.M. of $3x^4+8x^3+2x^2-8x-5$ and $6x^3+x^2-12x+5$.

(3.) What is the L.C.M. of $3x^3+8x^2y+3xy^2-2y^3$, $3x^3-7x^2y-7xy^2+3y^3$, and $2x^3-3x^2y-8xy^2-3y^3$?

101. Least Common Denominator.—The L.C.M. of a number of quantities is seldom required, except in reducing fractions to the same denominator, where it is in constant use, the L.C.M. of the denominators being the Least Common Denominator of the fractions.

L.C.D. may be written for Least Common Denominator.

Suppose we want to reduce to their L.C.D. the fractions $\frac{a}{bc}$, $\frac{b}{ac^3}$, and $\frac{c}{ab^2}$.

Here the L.C.M. of the denominators is ab^2c^3 (Art. 99); this, therefore, must form the denominator of each fraction.

That bc may become ab^2c^3, it will require to be multiplied by abc^2; and if the denominator be multiplied by this quantity, the numerator must also be multiplied by it, in order to preserve the value of the fraction unaltered (Art. 91).

$$\therefore \frac{a}{bc} = \frac{a}{bc} \times \frac{abc^2}{abc^2} = \frac{a^2bc^2}{ab^2c^3}.$$

Similarly ac^3 becomes ab^2c^3 by being multiplied by b^2, and

$$\therefore \frac{b}{ac^3} = \frac{b}{ac^3} \times \frac{b^2}{b^2} = \frac{b^3}{ab^2c^3}.$$

Also, ab^2 becomes ab^2c^3 by multiplication by c^3, and

$$\therefore \frac{c}{ab^2} = \frac{c}{ab^2} \times \frac{c^3}{c^3} = \frac{c^4}{ab^2c^3}.$$

The fractions $\frac{a}{bc}$, $\frac{b}{ac^3}$, and $\frac{c}{ab^2}$ have thus been reduced to their L.C.D. ab^2c^3, and have respectively become $\frac{a^2bc^2}{ab^2c^3}$, $\frac{b^3}{ab^2c^3}$, and $\frac{c^4}{ab^2c^3}$.

From a consideration of the method by which this has been effected, we easily deduce the following

RULE FOR THE REDUCTION OF FRACTIONS TO THEIR LEAST COMMON DENOMINATOR.

Find the L.C.M. of the denominators. This will form the L.C.D. Then multiply both upper and under lines of each fraction by such a quantity as will make its denominator equal to the one found.

Note.—The quantity by which the terms of any fraction must be multiplied is easily ascertained by dividing the L.C.D. of the whole by the denominator of that one.

Illustrative Examples.

(1.) Reduce the fractions $\frac{2}{3}$, $\frac{x}{4a}$, $\frac{a}{6b^2}$, and $\frac{y}{2a^2b}$ to equivalent ones having the least common denominator.

The L.C.M. of the denominators being $12a^2b^2$ (Art. 99), this is the L.C.D.

As $12a^2b^2 \div 3 = 4a^2b^2$, $\frac{2}{3} = \frac{2}{3} \times \frac{4a^2b^2}{4a^2b^2} = \frac{8a^2b^2}{12a^2b^2}$,

As $12a^2b^2 \div 4a = 3ab^2$, $\frac{x}{4a} = \frac{x}{4a} \times \frac{3ab^2}{3ab^2} = \frac{3ab^2x}{12a^2b^2}$,

As $12a^2b^2 \div 6b^2 = 2a^2$, $\frac{a}{6b^2} = \frac{a}{6b^2} \times \frac{2a^2}{2a^2} = \frac{2a^3}{12a^2b^2}$,

As $12a^2b^2 \div 2a^2b = 6b$, $\frac{y}{2a^2b} = \frac{y}{2a^2b} \times \frac{6b}{6b} = \frac{6by}{12a^2b^2}$.

The required fractions are therefore

$$\frac{8a^2b^2}{12a^2b^2}, \frac{3ab^2x}{12a^2b^2}, \frac{2a^3}{12a^2b^2}, \text{ and } \frac{6by}{12a^2b^2}.$$

(2.) Find in their least common denominator equivalent fractions for the following:—

$$\frac{1}{x-1}, \frac{x^2}{x^3-1}, \frac{x-1}{x^2-x+1}, \frac{x+1}{x^2+x+1}, \text{ and } \frac{x}{x^2-1}.$$

Re-writing with the denominators of the second and fifth fractions resolved into factors, we have

$$\frac{1}{x-1}, \frac{x^2}{(x-1)(x^2+x+1)}, \frac{x-1}{x^2-x+1}, \frac{x+1}{x^2+x+1}, \text{ and } \frac{x}{(x-1)(x+1)};$$

and each factor being taken once, we have the L.C.M. of the denominators, or

$$\begin{aligned} \text{L.C.D.} &= (x-1)(x^2+x+1)(x+1)(x^2-x+1) \\ &= (x^3-1)(x^3+1) = x^6-1. \end{aligned}$$

Now, the factors of x^6-1 that are not already contained in the denominator of the first fraction are

$$(x^2+x+1)(x^2-x+1)(x+1);$$

therefore $\frac{1}{x-1}$ becomes

$$\frac{(x^2+x+1)(x^2-x+1)(x+1)}{(x-1)(x^2+x+1)(x^2-x+1)(x+1)} \text{ or } \frac{x^5+x^4+x^3+x^2+x+1}{x^6-1}.$$

The factors of x^6-1 not already contained in the denominator x^3-1 are $(x+1)(x^2-x+1)$; therefore

$$\frac{x^2}{x^3-1} = \frac{x^2(x+1)(x^2-x+1)}{x^6-1} = \frac{x^5+x^2}{x^6-1}.$$

Similarly—

$$\frac{x-1}{x^2-x+1} = \frac{(x-1)(x-1)(x^2+x+1)(x+1)}{x^6-1} = \frac{x^5-x^3-x^2+1}{x^6-1};$$

so also—

$$\frac{x+1}{x^2+x+1} = \frac{(x+1)(x-1)(x^2-x+1)(x+1)}{x^6-1} = \frac{x^5-x^3+x^2-1}{x^6-1};$$

and $$\frac{x}{x^2-1} = \frac{x(x^2+x+1)(x^2-x+1)}{x^6-1} = \frac{x^5+x^3+x}{x^6-1}.$$

(3.) Reduce to their least common denominator the fractions $\frac{a+b}{4}$, $\frac{a}{a^2-4b^2}$, $\frac{b-a}{2b^2+ab-a^2}$, and $\frac{b}{a^3+a^2b-2ab^2}$.

Resolving the denominator into factors wherever possible, and changing the sign of all the terms of the third fraction, the above set of fractions becomes

$$\frac{a+b}{4}, \frac{a}{(a+2b)(a-2b)}, \frac{a-b}{(a+b)(a-2b)}, \text{ and } \frac{b}{a(a-b)(a+2b)},$$

in which the L.C.D. is plainly $4a(a+b)(a-b)(a+2b)(a-2b)$ or $4a(a^2-b^2)(a^2-4b^2)$.

Now, multiplying numerator and denominator of each

fraction by the factors which its denominator wants to bring it up to $4a(a^2-b^2)(a^2-4b^2)$, we have

$$\frac{a(a+b)(a^2-b^2)(a^2-4b^2)}{4a(a^2-b^2)(a^2-4b^2)}, \qquad \frac{4a^2(a^2-b^2)}{4a(a^2-b^2)(a^2-4b^2)},$$

$$\frac{4a(a-b)^2(a+2b)}{4a(a^2-b^2)(a^2-4b^2)}, \quad \text{and} \quad \frac{4b(a+b)(a-2b)}{4a(a^2-b^2)(a^2-4b^2)}.$$

In most cases it will be necessary to multiply out the terms of the numerators, but the denominators are generally better retained in factors.

EXAMPLES FOR PRACTICE.—XXXIV.

Reduce to their least common denominator each of the following sets of fractions:—

(1.) $\frac{a}{10}, \frac{b}{15}$.

(2.) $\frac{5x}{8}, \frac{7y}{12}, \frac{9z}{20}$.

(3.) $\frac{2}{ax}, \frac{3}{2bx^2}, \frac{4}{cx^3}$.

(4.) $\frac{a^2+b^2}{ab}, \frac{x^2-a^2}{ax}, \frac{x^2-b^2}{bx}$.

(5.) $\frac{a-b}{a+b}, \frac{a}{b}, \frac{b}{a}, \frac{a+b}{a-b}$.

(6.) $\frac{3}{(x-1)(x-2)}, \frac{2}{(x-1)(x-3)}, \frac{1}{(x-2)(x-3)}$.

(7.) $\frac{1}{x^2+x-12}, \frac{2}{x^2+2x-8}, \frac{2}{x^2+3x-4}, \frac{1}{x^2-5x+6}$.

(8.) $\frac{ax}{x^2+(2a-b)x-2ab}, \quad \frac{2bx}{x^2-(a+2b)x+2ab}, \quad \frac{3ab}{x^2+2(a-b)x-4ab}$

(9.) $\frac{x+1}{x^3+8}, \frac{1}{x^2+2x}, \frac{x-2}{x^2-2x+4}$.

(10.) $\frac{1}{x^3+ax+a^2}, \frac{1}{x^3-ax+a^2}, \frac{1}{x^4-a^2x^2+a^4}$.

(11.) $\frac{x+1}{6x^2+13x+6}, \frac{x}{9x^2-4}, \frac{1-x}{6-5x-6x^2}, 2x-3$.

(12.) $\frac{a^n}{b^3}, \frac{b^{n-1}}{(b+1)^n}, \frac{a+1}{b^3+b^2}, \frac{b+1}{a-1}, (a-1)^2(b+1)^3$.

CHAPTER VIII.

FRACTIONS.

(Continued.)

102. Reduction of Complex Fractions.—A fraction is ɔd to be *complex* when its numerator or denominator, or ɔh of them, contains a fraction. Thus—

$$\frac{\frac{a}{b}}{c}, \quad \frac{\frac{a}{x}+b}{\frac{a}{x}-\frac{b}{y}}, \quad \text{and} \quad \frac{x-\frac{ax}{a+x}}{a+\frac{ax}{a-x}} \quad \text{are complex fractions.}$$

Expressions of this character can be simplified by ıltiplying numerator and denominator by such a quan-y as will clear away the fractions from both upper and ıder lines.

In the above examples, if the numerator and denomin-or of the first be each multiplied by b, the expression ill become $\frac{a}{bc}$; of the second by xy, it will be equal to $\frac{(ax+bx)xy}{ay-bx}$; and of the third by a^2-x^2, it will be changed ıto $\frac{(ax+x^2-ax)(a-x)}{(a^2-ax+ax)(a+x)}$ or $\frac{x^2}{a^2}\cdot\frac{a-x}{a+x}$.

Notice that the multiplication of a fraction by its

denominator produces a quantity equal to its numerator; as $\frac{ax}{a+x} \times (a+x) = ax$.

103. In a way similar to the above, fractions of the form

$$\cfrac{a}{b+\cfrac{c}{d+\cfrac{e}{f+\cfrac{g}{h}}}}$$

may be simplified. Take the last complex fraction which appears in the expression, namely, $\cfrac{e}{f+\cfrac{g}{h}}$, and reduce it to the form $\frac{eh}{fh+g}$; the original fraction will now be $\cfrac{a}{b+\cfrac{c}{d+\cfrac{eh}{fh+g}}}$. Next reduce $\cfrac{c}{d+\cfrac{eh}{fh+g}}$ to $\frac{cfh+cg}{dfh+dg+eh}$, and we have $\cfrac{a}{b+\cfrac{cfh+cg}{dfh+dg+eh}}$; and finally, by similar reduction, this becomes $\frac{adfh+adg+aeh}{bdfh+bdg+beh+cfh+cg}$.

The work may be performed continuously; thus—

$$\cfrac{a}{b+\cfrac{c}{d+\cfrac{e}{f+\cfrac{g}{h}}}} = \cfrac{a}{b+\cfrac{c}{d+\cfrac{eh}{fh+g}}} = \cfrac{a}{b+\cfrac{cfh+cg}{dfh+dg+eh}} = \frac{a(dfh+dg+eh)}{bdfh+bdg+beh+cfh+cg}.$$

Fractions of the above character are called Continued Fractions.

EXAMPLES FOR PRACTICE.—XXXV.

Reduce the following to their simplest forms:—

(1.) $\frac{x+\frac{1}{3}}{x-\frac{1}{3}}$, $\frac{y-\frac{1}{2}}{y+\frac{1}{5}}$, and $\frac{1\frac{1}{4}-\frac{2}{3}x}{1\frac{1}{3}+\frac{1}{2}x}$.

(2.) $\frac{2a+1}{a^2-\frac{1}{4}}$, $\frac{\frac{1}{9}a^2-\frac{1}{4}x^2}{\frac{1}{2}x-\frac{1}{3}a}$, and $\frac{x+\frac{1}{y}}{y+\frac{1}{x}}$.

(3.) $\frac{a+\frac{x^2}{a-x}}{a^3+x^3}$ and $\frac{1-\frac{1}{x+1}}{1+\frac{1}{x-1}}$.

(4.) $\frac{\frac{x+y}{x-y}-\frac{x-y}{x+y}}{\frac{x-y}{x+y}+\frac{x+y}{x-y}}$ and $\frac{\frac{1}{x+y}-\frac{1}{y}}{\frac{1}{x}-\frac{1}{x-y}}$.

(5.) $\frac{1}{2+\frac{1}{3+\frac{1}{4}}}$ and $\frac{1}{a+\frac{1}{2b+\frac{1}{3a+\frac{1}{4b}}}}$.

(6.) $\frac{1}{x-\frac{1}{x+1+\frac{2}{x-1}}}$ and $\frac{1}{x+1+\frac{1}{x-2+\frac{1}{x+\frac{1}{x^2+x+1}}}}$.

104. Addition and Subtraction.—If a unit be divided into m parts, and $5a$ of these parts be taken, the quantity is expressed by the fraction $\frac{5a}{m}$ (Art. 90); and if $3a$ be

taken, by $\frac{3a}{m}$. Plainly, if first $5a$ and then $3a$ be taken, the total number taken will be $8a$; so that $\frac{5a}{m} + \frac{3a}{m} = \frac{8a}{m}$.

Similarly $\frac{x}{m} + \frac{y}{m} = \frac{x+y}{m}$.

If, after $5a$ parts have been taken, $3a$ of them be rejected, there will remain $2a$; so that $\frac{5a}{m} - \frac{3a}{m} = \frac{2a}{m}$.

Similarly $\frac{x}{m} - \frac{y}{m} = \frac{x-y}{m}$.

From this we see that fractions of the same denominator are added or subtracted by having their numerators added or subtracted.

Fractions having different denominators are of the nature of unlike quantities; and while they remain unlike, their sum or difference can only be indicated by signs (Art. 11). Thus, if it be required to take $\frac{z}{c}$ from the sum of $\frac{x}{a}$ and $\frac{y}{b}$, we can but indicate the result by writing $\frac{x}{a} + \frac{y}{b} - \frac{z}{c}$, so long as the denominators remain unlike. But fractions with different denominators can be reduced to a common denominator (Art. 101); and when this is done, their sum or difference can be represented as one fraction.

The L.C.D. in the above example being abc,

$$\frac{x}{a} + \frac{y}{b} - \frac{z}{c} = \frac{bcx}{abc} + \frac{acy}{abc} - \frac{abz}{abc} = \frac{bcx + acy - abz}{abc}.$$

Following

Illustrative Examples.

(1.) Add together $\frac{3a}{4}$, $\frac{4a}{5}$, and $\frac{5a}{6}$.

The L.C.M. of the denominators is $4 \times 5 \times 3$, or 60; and reducing to L.C.D., we have

$$\frac{3a}{4} + \frac{4a}{5} + \frac{5a}{6} = \frac{45a}{60} + \frac{48a}{60} + \frac{50a}{60} = \frac{143a}{60}.$$

(2.) Take $\frac{5bc}{6mn}$ from $\frac{3ac}{4m^2}$.

The L.C.D. $= 3 \times 4 \times m^2 \times n = 12m^2n$.

$$\therefore \frac{3ac}{4m^2} - \frac{5bc}{6mn} = \frac{9acn}{12m^2n} - \frac{10bcm}{12m^2n} = \frac{9acn - 10bcm}{12m^2n}.$$

(3.) Find the sum of $\frac{a-2b}{a+2b}$, $\frac{a+2b}{a-2b}$, and $\frac{a^2}{2b^2}$.

The L.C.D. $= 2b^2(a^2 - 4b^2)$.

$$\therefore \text{the sum} = \frac{2b^2(a-2b)^2 + 2b^2(a+2b)^2 + a^2(a^2-4b^2)}{2b^2(a^2-4b^2)}$$

$$= \frac{2a^2b^2 - 8ab^3 + 8b^4 + 2a^2b^2 + 8ab^3 + 8b^4 + a^4 - 4a^2b^2}{2b^2(a^2-4b^2)}$$

$$= \frac{a^4 + 16b^4}{2b^2(a^2-4b^2)}.$$

(4.) Take $\frac{3(a+x)x^2}{a^2+ax-2x^2}$ from the sum of $a-x$ and $\frac{2ax}{a-x}$.

$$a - x + \frac{2ax}{a-x} = \frac{a^2 - 2ax + x^2 + 2ax}{a-x} = \frac{a^2+x^2}{a-x}; \text{ and}$$

$$\frac{a^2+x^2}{a-x} - \frac{3(a+x)x^2}{a^2+ax-2x^2} = \frac{a^2+x^2}{a-x} - \frac{3ax^2+3x^3}{(a+2x)(a-x)}$$

$$= \frac{(a^2+x^2)(a+2x) - (3ax^2+3x^3)}{(a+2x)(a-x)}$$

$$= \frac{a^3 + 2a^2x + ax^2 + 2x^3 - 3ax^2 - 3x^3}{(a+2x)(a-x)}$$

$$= \frac{a^3 - x^3 + 2a^2x - 2ax^2}{(a+2x)(a-x)} = \frac{(a^2+ax+x^2)(a-x)+2ax(a-x)}{(a+2x)(a-x)}$$

$$= \frac{(a^2+3ax+x^2)(a-x)}{(a+2x)(a-x)} = \frac{a^2+3ax+x^2}{a+2x} = a+x-\frac{x^2}{a+2x}.$$

(5.) Simplify $\frac{a}{(a-b)(a-c)} + \frac{b}{(b-a)(b-c)} + \frac{c}{(c-a)(c-b)}$.

Here, as no two factors in the denominator seem the same, a very natural mistake would be to take the product of the whole six for the L.C.D.; but it should be observed that $b-a = -(a-b)$, $c-a = -(a-c)$, and $c-b = -(b-c)$; wherefore $(b-a)(b-c) = -(a-b)(b-c)$, and $(c-a)(c-b) = (a-c)(b-c)$, by Art. 95; and thus the whole expression can be written

$$\frac{a}{(a-b)(a-c)} - \frac{b}{(a-b)(b-c)} + \frac{c}{(a-c)(b-c)};$$

in which the L.C.D. is $(a-b)(a-c)(b-c)$.

$$\therefore \frac{a}{(a-b)(a-c)} - \frac{b}{(a-b)(b-c)} + \frac{c}{(a-c)(b-c)}$$

$$= \frac{a(b-c)-b(a-c)+c(a-b)}{(a-b)(a-c)(b-c)} = \frac{ab-ac-ab+bc+ac-bc}{(a-b)(a-c)(b-c)} = \quad = 0.$$

Had the numerators in this question been squares, it would have been simplified most neatly in the following manner :—

$$\frac{a^2}{(a-b)(a-c)} - \frac{b^2}{(a-b)(b-c)} + \frac{c^2}{(a-c)(b-c)}$$

$$= \frac{a^2(b-c)-b^2(a-c)+c^2(a-b)}{(a-b)(a-c)(b-c)} = \frac{a^2b-a^2c-ab^2+b^2c+c^2(a-b)}{(a-b)(a-c)(b-c)}$$

$$= \frac{ab(a-b)-c(a^2-b^2)+c^2(a-b)}{(a-b)(a-c)(b-c)} = \frac{ab-c(a+b)+c^2}{(a-c)(b-c)}$$

$$= \frac{ab-ac-bc+c^2}{(a-c)(b-c)} = \frac{(a-c)(b-c)}{(a-c)(b-c)} = 1.$$

(6.) Reduce

$$\left(1 - \frac{a+x}{a-x}\right)\left(\frac{1}{1 - \frac{a^2+x^2}{a^2-x^2}}\right)\left(\frac{1}{1 - \frac{1}{1 - \frac{a^3+x^3}{a^3-x^3}}}\right)$$

to its simplest form.

Simplify each of the factors by one or other of the previous rules:—

$$1 - \frac{a+x}{a-x} = \frac{a-x-a-x}{a-x} = \frac{-2x}{a-x}.$$

$$\frac{1}{1 - \frac{a^2+x^2}{a^2-x^2}} = \frac{a^2-x^2}{a^2-x^2-a^2-x^2} = \frac{a^2-x^2}{-2x^2}.$$

$$\frac{1}{1 - \frac{1}{1 - \frac{a^3+x^3}{a^3-x^3}}} = \frac{1}{1 - \frac{a^3-x^3}{a^3-x^3-a^3-x^3}} = \frac{1}{1 - \frac{a^3-x^3}{-2x^3}}$$

$$= \frac{1}{1 + \frac{a^3-x^3}{2x^3}} = \frac{2x^3}{2x^3+a^3-x^3} = \frac{2x^3}{a^3+x^3}.$$

The expression therefore becomes—

$$\frac{-2x}{a-x} \times \frac{a^2-x^2}{-2x^2} \times \frac{2x^3}{a^3+x^3} = \frac{2x^2}{a^2-ax+x^2}.$$

EXAMPLES FOR PRACTICE.—XXXVI.

Find the simplest values of the following:—

(1.) $\frac{11a}{12} + \frac{7a}{8} + \frac{5a}{9}$.

(2.) $\frac{ab^2c}{5} - \frac{ab^2c}{7}$.

(3.) $\frac{2}{bc} + \frac{3}{ac} + \frac{2}{ab}$.

(4.) $\frac{a}{bx} - \frac{x}{ab}$.

(5.) $\frac{x}{x-a} + \frac{x}{a}$.

(6.) $1 - \frac{x}{x+a}$.

(7.) $\dfrac{a+x}{x}+\dfrac{x}{a+x}-\dfrac{a^2}{ax+x^2}$.

(8.) $\dfrac{x}{x+y}+\dfrac{y}{x-y}+\dfrac{2xy}{x^2-y^2}$.

(9.) $\dfrac{x}{x-y}+\dfrac{y}{y-x}$.

(10.) $2-\left(\dfrac{a-b}{a}+\dfrac{a}{a-b}\right)$.

(11.) $\dfrac{1}{x^2-4}-\dfrac{1}{x^2+x-2}$.

(12.) $\dfrac{1}{4}+\dfrac{x+1}{4x-2}-\dfrac{2x^2+1}{4x^2-1}$.

(13.) $1+\dfrac{1}{x-1}+\dfrac{x}{x+1}$.

(14.) $\dfrac{y}{x}-\dfrac{x}{x-y}+\dfrac{y^3}{(x^2-y^2)x}$

(15.) $\dfrac{1}{(x+1)(x+2)}+\dfrac{1}{(x+1)(x-3)}+\dfrac{1}{(x+2)(x-3)}$

(16.) $\dfrac{1}{(x+1)(x+2)}-\dfrac{2}{(x+1)(x+3)}+\dfrac{1}{(x+2)(x+3)}$

(17.) $\dfrac{1}{x-3}-\dfrac{x-3}{x^2+3x+9}$.

(18.) $\dfrac{x-2}{x^2-2x+4}+\dfrac{x+2}{x^2+2x+4}$.

(19.) $\dfrac{2x^2+2a^2}{x^2-a^2}-\dfrac{x^2-ax+a^2}{x^2+ax+a^2}-\dfrac{x^2+ax+a^2}{x^2-ax+a^2}$.

(20.) $\dfrac{x-1}{x^2-2x+2}+\dfrac{x+1}{x^2+2x+2}+\dfrac{2x^3}{x^4-4}$.

(21.) $\dfrac{1}{4x+1}-\left\{\dfrac{1}{3x+1}-\left(\dfrac{1}{1-4x}-\dfrac{1}{1-3x}\right)\right\}$.

(22.) $\left(1-\dfrac{1}{x^2}\right)\left(1-\dfrac{1}{x-2}\right)\left(1+\dfrac{1}{x-3}\right)\left(1+\dfrac{1}{x^2-1}\right)$

(23.) $\left(\dfrac{a}{b}+\dfrac{b}{a}+\dfrac{a^2+b^2}{ab}\right)\left(\dfrac{x}{y}+\dfrac{y}{x}-\dfrac{x^2+y^2}{xy}\right)$.

(24.) $\dfrac{\dfrac{a}{a+b}}{\dfrac{a+b}{a}}-\dfrac{\dfrac{b}{a+b}}{\dfrac{a+b}{b}}$.

(25.) $\left(\frac{x}{x+y} - \frac{y}{x-y}\right) - \left(\frac{x}{x-y} - \frac{y}{x+y}\right)$.

(26.) $\frac{2\frac{1}{4} - x^2}{x + 1\frac{1}{2}} - \frac{\frac{1}{2} - 2x^2}{1 - 2x}$.

(27.) $\frac{b+c}{(a-b)(a-c)} + \frac{a+c}{(b-a)(b-c)} + \frac{a+b}{(c-a)(c-b)}$.

(28.) $1 - \dfrac{1 - \dfrac{a^2+b^2}{2ab}}{1 - \dfrac{2ab}{a^2+b^2}}$

(29.) $\left(1 - \frac{x^2+y^2}{2xy}\right)\left(1 - \frac{y^2+z^2}{2yz}\right)\left(\frac{x^2+z^2}{xz} - 2\right)$.

(30.) $\left\{\frac{a+x}{x} - \frac{a}{a+x} - \frac{(a-x)^2}{2ax} - \frac{a^2-2x^2}{a^2-x^2}\right\} \div \frac{1}{2}\left(\frac{1}{x} + \frac{x}{a^2-ax}\right)$.

(31.) $\frac{2yz}{x(y+z-x)} - \left\{\left(1 + \frac{x^2+y^2-z^2}{2xy}\right) \div \left(1 - \frac{x^2-y^2-z^2}{2yz}\right)\right\}$.

(32.) $\dfrac{x}{x - \dfrac{1}{y + \dfrac{1}{x - \dfrac{1}{y}}}} - \dfrac{1}{1 + \dfrac{1}{xy - \dfrac{1}{1 + \dfrac{1}{xy}}}}$.

(33.) $\left(\dfrac{1}{1 + \dfrac{1}{x - 1 + \dfrac{1}{x+1}}}\right)\left(\dfrac{1}{1 + \dfrac{1}{x + \dfrac{1}{x-1}}}\right) + \dfrac{2}{x + 1 + \dfrac{1}{x}}$.

(34.) $\left\{\left(\frac{a}{b} + \frac{b}{a}\right)^2 + \left(\frac{b}{c} - \frac{c}{b}\right)^2\right\} \div \left(\frac{a}{c} + \frac{c}{a}\right)^2$

(35.) $\frac{a^3}{(a-b)(a-c)} + \frac{b^3}{(b-a)(b-c)} + \frac{c^3}{(c-a)(c-b)}$.

(36.) $\frac{a^n+b^n}{a^nb^n}(a^{2n}+b^{2n}-c^{2n}) + \frac{b^n+c^n}{b^nc^n}(b^{2n}+c^{2n}-a^{2n}) + \frac{c^n+a^n}{c^na^n}(c^{2n}+a^{2n}-b^{2n})$.

105. Substitutions.

Illustrative Examples.

(1.) Find the numerical value of—

$\frac{a^2-ax+x^2}{a^2+ax+x^2} - \frac{a-x}{a+x}\cdot\frac{a}{x}$, when $a=\frac{1}{3}$ and $x=\frac{1}{4}$.

$$\frac{a^2-ax+x^2}{a^2+ax+x^2} - \frac{a-x}{a+x}\cdot\frac{a}{x} = \frac{\frac{1}{9}-\frac{1}{12}+\frac{1}{16}}{\frac{1}{9}+\frac{1}{12}+\frac{1}{16}} - \frac{\frac{1}{3}-\frac{1}{4}}{\frac{1}{3}+\frac{1}{4}}\times\frac{\frac{1}{3}}{\frac{1}{4}}$$

$$= \frac{16-12+9}{16+12+9} - \frac{4-3}{4+3}\times\frac{4}{3} = \frac{13}{37} - \frac{1}{7}\times\frac{4}{3}.$$

$$= \frac{13}{37} - \frac{4}{21} = \frac{273-148}{777} = \frac{125}{777}.$$

(2.) What is the numerical value of—

$$\frac{1}{x+1+\dfrac{1}{3x+1+\dfrac{1}{8x}}}, \text{ when } x=2?$$

$$\frac{1}{x+1+\dfrac{1}{3x+1+\dfrac{1}{8x}}} = \frac{1}{3+\dfrac{1}{7+\frac{1}{16}}} = \frac{1}{3+\frac{16}{113}} = \frac{113}{355}.$$

Or, the algebraical expression may be simplified before the substitution is made—

$$\frac{1}{x+1+\dfrac{1}{3x+1+\dfrac{1}{8x}}} = \frac{1}{x+1+\dfrac{8x}{24x^2+8x+1}}$$

$$= \frac{24x^2+8x+1}{24x^3+32x^2+17x+1} = \frac{96+16+1}{192+128+34+1} = \frac{113}{355}.$$

(3.) Find the simplest form of $\frac{x}{y}+\frac{x-1}{y+1}$ when $x=\frac{a}{a+b}$ and $y=\frac{b}{a-b}$.

$$\frac{x}{y}+\frac{x-1}{y+1}=\frac{\frac{a}{a+b}}{\frac{b}{a-b}}+\frac{\frac{a}{a+b}-1}{\frac{b}{a-b}+1}=\frac{a(a-b)}{b(a+b)}+\frac{(a-a-b)(a-b)}{(b+a-b)(a+b)}$$

$$=\frac{a(a-b)}{b(a+b)}+\frac{-b(a-b)}{a(a+b)}=\left(\frac{a}{b}-\frac{b}{a}\right)\cdot\frac{a-b}{a+b}$$

$$=\frac{a^2-b^2}{ab}\cdot\frac{a-b}{a+b}=\frac{(a-b)^2}{ab}.$$

EXAMPLES FOR PRACTICE.—XXXVII.

Find the simplest forms of the following expressions:—

(1.) $\frac{3a}{b+c}-\frac{2b}{a-c}+\frac{c}{a+b}$ when $a=8$, $b=6$, and $c=3$.

(2.) $\frac{a^3+2a^2x+3ax^2}{a^2-ax+x^2}$ when $a=\frac{1}{2}$ and $x=\frac{1}{5}$.

(3.) $\frac{(x+2y)z+(y+2z)x+(z+2x)y}{(x+3y)z-(y+3z)x+(z+3x)y}$ when $x=3y=4z$.

(4.) $\frac{(x-a)(x-2a)(x-3b)(x-4b)}{4a^2b^2}$ when $x=2a$.

(5.) $\frac{(x-a)(x-2a)(x-3b)(x-4b)}{4a^2b^2}$ when $x=2b$.

(6.) $\frac{a-x}{1+ax}+\frac{b+x}{1-bx}$ when $x=\frac{a-b}{1+ab}$.

(7.) $\left(x^2-\frac{1}{x^2}\right)\left(xy-\frac{1}{xy}\right)\left(y^2-\frac{1}{y^2}\right)$ when $x=3\frac{1}{3}$ and $y=\frac{3}{10}$.

(8.) $\cfrac{1}{x+\cfrac{1}{x-1+\cfrac{1}{x-2+\cfrac{1}{x-3}}}}$ when $x=4$, $x=3$, $x=2$, $x=1$ successively.

(9.) $\dfrac{1+2x}{1+y} - \dfrac{x}{y}$ when $x = \dfrac{a-1}{a(b-2)}$ and $y = \dfrac{a-2}{a(b-1)}$.

(10.) $\dfrac{a-x}{b+y} \div \dfrac{b-y}{a+x}$ when $x = \dfrac{a^2+b^2-c^2}{2b}$ and $y = \dfrac{a^2-b^2-c^2}{2c}$.

(11.) $\cfrac{\cfrac{4bc}{a^2}}{\left(\cfrac{b+c}{a}\right)^2 - 1} - 1$ when $\frac{1}{2}(a+b+c) = s$.

(12.) $\dfrac{a(b+y)+b(a+x)+c(x+y)}{a+b}$ when $x = \dfrac{a^2+bc}{b+c}$ and $y = \dfrac{b^2+ac}{a+c}$.

106. Occasionally when the required substitution is made, the numerator or the denominator, or both, may become *nothing*. For instance, in the expressions $\dfrac{a-x}{a+x}$, $\dfrac{a^2+x^2}{ax-x^2}$, and $\dfrac{a^2-ax}{a^2-x^2}$, if a be substituted for x, the first becomes $\dfrac{0}{2a}$, the second $\dfrac{2a^2}{0}$, and the third $\dfrac{0}{0}$.

We easily perceive that $\dfrac{0}{2a}$ is equal to 0, for the $\dfrac{1}{2a}$th part of nothing must still be nothing; but it will require a little examination to ascertain what are represented by $\dfrac{2a^2}{0}$ and $\dfrac{0}{0}$.

107. When one number is divided by another, the quotient is less than, equal to, or greater than unity,

according as the divisor is greater than, equal to, or less than the dividend; thus, $8 \div 12 = \frac{8}{12}$ or $\frac{2}{3}$, $8 \div 8 = \frac{8}{8}$ or 1, and $8 \div 4 = \frac{8}{4}$ or 2; and the smaller the divisor, the greater the quotient, as the following examples show:—$8 \div 2 = 4$, $8 \div \frac{1}{2} = 16$, $8 \div \frac{1}{1000} = 8000$, $8 \div \frac{1}{10^{10}} = 80000000000$, and $8 \div \frac{1}{10^{10^{10}}} = 8$ followed by ten thousand million ciphers, a number great beyond all human powers of conception.

As the divisor continually grows less and less, it more and more nearly becomes equal to nothing, and 0 may be put to represent its ultimate value; while the quotient, growing constantly greater and greater, becomes larger than any quantity that can be named, and is said to be infinite. The character ∞ is used to indicate infinity. Any expression, therefore, of the form $\frac{x}{0}$ means the division of a finite quantity by one infinitely small, and as the quotient is infinitely great, it is represented by ∞.

$$\text{So that } \frac{2a^2}{0} = \infty \text{ (read } \textit{infinity}\text{).}$$

That a finite quantity divided by 0 becomes infinite may be shown in another way; for if $\frac{1}{1-x}$ be expanded into a series (Art. 72, Ex. 4), it becomes $1 + x + x^2 + x^3 + x^4 + x^5 +$ etc. to infinity, and if x be made equal to 1, we have

$$\frac{1}{1-1} = \frac{1}{0} = 1 + 1 + 1 + 1 + 1 + \text{ etc. to infinity,}$$

$$\therefore \frac{1}{0} = \infty .$$

We may remark that although $\frac{1}{0}$, $\frac{x}{0}$, and $\frac{2a^2}{0}$ are all equal to ∞, they cannot be considered equal to one another.

108. In investigating the meaning of the expression $\frac{0}{0}$, we must remember that any quantity multiplied by 0 becomes 0. If, therefore, we multiply both terms of a fraction, as, say, $\frac{2}{3}$, by 0, we will have $\frac{2}{3} = \frac{2 \times 0}{3 \times 0} = \frac{0}{0}$. Conversely, if we could resolve each of the terms of this $\frac{0}{0}$ into what may be called its factors, we should have $\frac{0}{0} = \frac{0 \times 2}{0 \times 3} = \frac{2}{3}$ by dividing out $\frac{0}{0}$.

In the fraction $\frac{a^2 - ax}{a^2 - x^2}$, $(x = a)$, given above, if we make the substitution at once, we get $\frac{0}{0}$; but if we first resolve each line into factors, and then substitute, we have $\frac{a(a-a)}{(a+a)(a-a)}$ or $\frac{a \times 0}{2a \times 0}$, a quantity equal to $\frac{0}{0}$ if we perform the multiplication indicated, but equal to $\frac{1}{2}$ when the common factors a and 0 are cancelled.

So in the expression $\frac{x^2 - 1}{x^3 + 1}$, for $x = -1$, we get $\frac{0}{0}$, unless the common factor $x + 1 = 0$ be divided out, when we have $\frac{x^2 - 1}{x^3 + 1} = \frac{x - 1}{x^2 - x + 1} = \frac{-1 - 1}{1 + 1 + 1} = \frac{-2}{3}$.

Generally, therefore, when it is found that a fraction becomes $\frac{0}{0}$, we must resolve numerator and denominator into factors, and cancel those common to both, before making the substitution.

Fractions of the form $\frac{0}{0}$ are called Vanishing Fractions.

Find the value of the fraction $\frac{6x^2 - 13x + 6}{9x^2 - 12x + 4}$ when $x = \frac{2}{3}$.

$$\frac{6x^2 - 13x + 6}{9x^2 - 12x + 4} = \frac{6 \times \frac{4}{9} - 13 \times \frac{2}{3} + 6}{9 \times \frac{4}{9} - 12 \times \frac{2}{3} + 4} = \frac{24 - 78 + 54}{36 - 72 + 36} = \frac{0}{0},$$

a vanishing fraction.

Since $x=\frac{2}{3}$, $x-\frac{2}{3}=0$, and $x-\frac{2}{3}$ or $3x-2$ must be a common factor of numerator and denominator. Dividing this out of each, we have—

$$\frac{2x-3}{3x-2}=\frac{2\times\frac{2}{3}-3}{3\times\frac{2}{3}-2}=\frac{4-9}{6-6}=\frac{-5}{0}=-\infty.$$

EXAMPLES FOR PRACTICE.—XXXVIII.

Show that the following are vanishing fractions, and find the value of each :—

(1.) $\frac{x^2-4}{x^3-8}$ when $x=2$.

(2.) $\frac{x^2-8x+16}{x^3-5x^2+4x}$ when $x=4$.

(3.) $\frac{e^x-e^{-x}}{e^{2x}-1}$ when $x=0$.

(4.) $\frac{2+3x-x^3}{3+5x+x^2-x^3}$ when $x=-1$.

(5.) $\frac{x-y}{x+y}+\frac{4x^2}{x^2-y^2}-\frac{x+y}{x-y}$ when $x=y$.

(6.) $\frac{a^3x^3-a^2bx^2-ab^2x+b^3}{a^3x^3-3a^2bx^2+3ab^2x-b^3}$ when $x=\frac{b}{a}$.

109. Theorems.—It is worthy of remark that the numerators and denominators of two equal fractions may be combined in a variety of ways by the processes of addition and subtraction.

Let $\frac{a}{b}=\frac{c}{d}$.

Then $\frac{a}{b}+1=\frac{c}{d}+1$; or, $\frac{a+b}{b}=\frac{c+d}{d}$. I.

So also, $\frac{a}{b}-1=\frac{c}{d}-1$; or, $\frac{a-b}{b}=\frac{c-d}{d}$. II.

And dividing I. by II., we have—

$$\frac{a+b}{b} \times \frac{b}{a-b} = \frac{c+d}{d} \times \frac{d}{c-d}; \text{ or, } \frac{a+b}{a-b} = \frac{c+d}{c-d}. \quad \text{III.}$$

Also, dividing II. by I. we get—

$$\frac{a-b}{b} \times \frac{b}{a+b} = \frac{c-d}{d} \times \frac{d}{c+d}; \text{ or, } \frac{a-b}{a+b} = \frac{c-d}{c+d}. \quad \text{IV.}$$

$$\text{Since } \frac{a}{b} = \frac{c}{d}, \quad \frac{a}{b} \times \frac{b}{c} = \frac{c}{d} \times \frac{b}{c}; \text{ or, } \frac{a}{c} = \frac{b}{d}. \quad \text{V.}$$

And therefore, as above,—

$$\frac{a+c}{a-c} = \frac{b+d}{b-d}, \text{ and } \frac{a-c}{a+c} = \frac{b-d}{b+d}. \quad \text{VI.}$$

Again, since $\frac{a}{b} = \frac{c}{d}$, $ad = bc$.

And adding or subtracting cd to or from each side, we have—

$$ad \pm cd = bc \pm cd, \text{ or } (a \pm c)d = (b \pm d)c;$$

$$\text{and } \therefore \frac{a \pm c}{b \pm d} = \frac{c}{d} \text{ or } \frac{a}{b}. \quad \text{VII.}$$

So that the sum of the numerators upon the sum of the denominators, or the difference of the numerators upon the difference of the denominators, is equal to either of the original fractions.

These theorems are frequently of use in the solution of fractional equations.

They may also be still further extended; for if

$$\frac{a}{b} = \frac{c}{d} = x,$$

Then $a = bx$, and $c = dx$;

Also, $ma = mbx$, and $nc = ndx$.

From which, $ma \pm nc = mbx \pm ndx = (mb \pm nd)x$.

$$\text{And } \therefore \frac{ma \pm nc}{mb \pm nd} = x = \frac{a}{b} = \frac{c}{d} \quad \text{VIII.}$$

Similarly, if $\frac{a}{b} = \frac{c}{d} = \frac{e}{f}$, it may be shown that

$$\frac{ma+nc+pe}{mb+nd+pf} = \frac{a}{b}, \text{ etc.} \qquad \text{IX.}$$

Likewise, if $\frac{a}{b}$ be greater than $\frac{c}{d}$, and $\frac{c}{d}$ greater than $\frac{e}{f}$, then $\frac{a+c+e}{b+d+f}$ is less than $\frac{a}{b}$ and greater than $\frac{e}{f}$.

Let $>$ stand for greater than, and $<$ for less than.

Put $\frac{a}{b} = y$, then $\frac{c}{d} < y$, and $\frac{e}{f} < y$,

Also, $a = by$, $c < dy$, and $e < fy$,

And, $a+c+e < (b+d+f)y$;

$$\therefore \frac{a+c+e}{b+d+f} < y \text{ or } \frac{a}{b}.$$

Next put $\frac{e}{f} = z$, then $\frac{c}{d} > z$, and $\frac{a}{b} > z$,

Also, $e = fz$, $c > dz$, $a > bz$,

And, $a+c+e > (b+d+f)z$;

$$\therefore \frac{a+c+e}{b+d+f} > z \text{ or } \frac{e}{f}. \qquad \text{X.}$$

110. Fractional Equations.

(1.) Find the value of x in the equation

$$\frac{5x}{3} - \frac{4}{5} - \frac{3x-2}{4} = 7 - \frac{4x+3}{15}.$$

In solving an equation of this kind, the first step would appear to be to bring all the fractions to the same denominator. Let this be done, and we have—

$$\frac{100x}{60} - \frac{48}{60} - \frac{45x-30}{60} = \frac{420}{60} - \frac{16x+12}{60}.$$

Or, $$\frac{100x - 48 - 45x + 30}{60} = \frac{420 - 16x - 12}{60}.$$

But plainly, when two equal fractions have their denom-

inators equal, their numerators must be equal also; and therefore $100x - 48 - 45x + 30 = 420 - 16x - 12$, which, by the ordinary process of solution, gives $71x = 426$ and $x = 6$.

Observe that removing the denominator is equivalent to multiplying each side of the equation by 60, and that this multiplication could have been at once applied to each of the terms as they stood at first.

(2.) Solve the equation

$$\cdot 2x - \frac{\cdot 75 - 2x}{\cdot 5} = \frac{3x - 7}{\cdot 4} + \frac{\cdot 35 - \cdot 08x}{\cdot 1}.$$

Remove the decimals by multiplying the first quantity on each side by $\frac{10}{10}$, and the second by $\frac{100}{100}$; this gives—

$$\frac{2x}{10} - \frac{75 - 200x}{50} = \frac{30x - 70}{4} + \frac{35 - 8x}{10}.$$

Now cancel as far as possible, and we will have—

$$\frac{x}{5} - \frac{3 - 8x}{2} = \frac{15x - 35}{2} + \frac{35 - 8x}{10}.$$

Next clear off fractions by using L.C.M. of denominators as multiplier—

$$2x - 15 + 40x = 75x - 175 + 35 - 8x,$$
$$50x - 75x = 50 - 175,$$
$$25x = 125, \quad \text{and } \therefore\ x = 5.$$

(3.) Find the value of x in the following equation—

$$\frac{a - x}{b} + \frac{b + x}{a} = 2.$$

Multiplying by ab, $\quad a^2 - ax + b^2 + bx = 2ab$,
Trans. and collecting, $(a - b)x = a^2 - 2ab + b^2 = (a - b)^2$,
Dividing out $a - b$, $\quad x = a - b$.

(4.) Find the value of x in the equation

$$\frac{2x + 1}{x} - \frac{4x - 3}{x^2 + x} = \frac{2x - 5}{x - 1}.$$

The L.C.D. being $x(x + 1)(x - 1)$, the fractions may be

cleared away, by multiplying at once by this quantity, or successively by its factors. Taking the latter method, and multiplying first by x, we have—

$$2x+1-\frac{4x-3}{x+1}=\frac{2x^2-5x}{x-1}.$$

Multi. by $x-1$, $2x^2-x-1=\dfrac{4x^2-7x+3}{x+1}=2x^2-5x$,

Collecting, $4x-1=\dfrac{4x^2-7x+3}{x+1}$,

Multi. by $x+1$, $4x^2+3x-1=4x^2-7x+3$,

Collecting, $10x=4$, and $x=\dfrac{2}{5}$.

(5.) Solve the equation—

$$\frac{x-3}{x-5}-\frac{x-7}{x-9}=\frac{x-2}{x-4}-\frac{x-6}{x-8}.$$

Reduce to common denominators the two sides separately, then—

$$\frac{x^2-12x+27-(x^2-12x+35)}{(x-5)(x-9)}=\frac{x^2-10x+16-(x^2-10x+24)}{(x-4)(x-8)}$$

and $$\frac{-8}{x^2-14x+45}=\frac{-8}{x^2-12x+32}.$$

As the numerators are equal, the denominators must also be equal, and therefore—

$$x^2-14x+45=x^2-12x+32;$$

from which $2x=13$, and $x=6\frac{1}{2}$.

(6.) Find the value of x which satisfies the equation—

$$\frac{ax-m-n}{bx-m+p}=\frac{ax-p-q}{bx+n-q}.$$

Clearing away fractions, as before, we get—

$$abx^2+anx-aqx-bmx-mn+mq-bnx-n^2+nq$$
$$=abx^2-amx+apx-bpx+mp-p^2-bqx+mq-pq,$$

Collecting, $(am+an-ap-aq-bm-bn+bp+bq)x$
$$= mn+mp+n^2-p^2-nq-pq,$$
Factoring, $(a-b)(m+n-p-q)x=(n+p)(m+n-p-q)$,
Cancelling, $(a-b)x=n+p.$
$$\therefore\ x=\frac{n+p}{a-b}.$$

Questions of this form may frequently be solved by the help of the theorem proved in Art. 109, VII.,—namely, if $\frac{a}{b}=\frac{c}{d}$, then $\frac{a-c}{b-d}$ is equal to either of the fractions.

Subtracting, therefore, $ax-p-q$ from $ax-m-n$ and $bx+n-q$ from $bx-m+p$, we have—
$$\frac{ax-m-n}{bx-m+p}=\frac{-m-n+p+q}{-m-n+p+q}=1,$$
which gives $ax-m-n=bx-m+p$
$$(a-b)x=n+p,\quad \text{and}\ \therefore\ x=\frac{n+p}{a-b}.$$

(7.) Find x in the following—
$$\frac{4x^2+6x-8}{6x^2+2x+9}=\frac{2x^2+3x-5}{3x^2+x+3}.$$

In order to avail ourselves of the artifice used in last question, we will here require to multiply the second side of the equation by $\frac{2}{2}$. This gives—
$$\frac{4x^2+6x-8}{6x^2+2x+9}=\frac{4x^2+6x-10}{6x^2+2x+6}=\frac{2}{3}.\quad \text{(Art. 109, VII.)}$$
$$\therefore\ 12x^2+18x-24=12x^2+4x+18$$
$$14x=42,\quad \text{and}\ x=3.$$

(8.) What value has x in the following—
$$\frac{4ax+(a+b)^2}{4ax-(a+b)^2}=\frac{x-a+b}{x-a-b}?$$

By Art. 109, III., if $\frac{a}{b}=\frac{c}{d}$, then $\frac{a+b}{a-b}=\frac{c+d}{c-d}$.

ring this here, we have—

$$\frac{8ax}{2(a+b)^2} = \frac{2(x-a)}{2b} \text{ or } \frac{4ax}{(a+b)^2} = \frac{x-a}{b};$$

$$\frac{x}{(a+b)^2} = \frac{x-a}{4ab} = \frac{a}{(a-b)^2} \text{ by Theorem VII., Art. 109.}$$

$$\therefore\ x = \frac{a(a+b)^2}{(a-b)^2}.$$

EXAMPLES FOR PRACTICE.—XXXIX.

ıd the value of the unknown quantity in each of the ring equations :—

(1.) $\frac{x}{4} + \frac{x}{6} + \frac{x}{12} = 1.$

(2.) $\frac{3x}{4} - \frac{4x}{5} + \frac{2x}{15} = \frac{1}{2}.$

(3.) $\frac{x}{5} + \frac{x}{3} - 2\frac{1}{2} = \frac{5x}{12} - \frac{3}{4}.$

(4.) $\frac{4x+1}{3} - 3x = \frac{5-9x}{4} + 2.$

(5.) $\frac{3x+1}{4} - \frac{5x-2}{8} = \frac{7x-1}{3} - \frac{11x+4}{6}.$

(6.) $\frac{x-7}{4} - \frac{x+4}{7} = \frac{x+5}{8} - \frac{9-x}{6} - 2.$

(7.) $\frac{4x-1\frac{1}{2}}{4\frac{1}{2}} + 3 = \frac{2}{9}(x+3).$

(8.) $\frac{3x-9\frac{3}{4}}{5} - \frac{4x+5\frac{1}{3}}{25} = \frac{\frac{1}{3}x-\frac{1}{2}}{2}.$

(9.) $\frac{2x-5\frac{1}{2}}{7} + \frac{\frac{3}{4}x+4}{2\frac{1}{2}} = \frac{\frac{2}{5}x-\frac{3}{4}}{\frac{3}{2}-\frac{2}{5}}.$

(10.) $\cdot 2x - 3 = \cdot 5x - \cdot 03x - 5\cdot 7.$

(11.) $\cdot 12x + \cdot 8(x+3) = 10 - \cdot 2(5x - \cdot 4)$.

(12.) $\frac{2}{5}(4x - 2\cdot 5) + \cdot 1x = 7\cdot 75 - \frac{3}{4}(\cdot 4x + 5)$.

(13.) $\frac{1}{x} - \frac{1}{2x} + \frac{2}{3x} - \frac{3}{4x} = \frac{5}{8}$.

(14.) $\frac{3x-5}{x} - \frac{5x+3}{4x} = \frac{1}{12x} + \frac{7}{24}$.

(15.) $3x - \frac{5x-2}{8x} = \frac{4x^2+3}{2x} - \frac{5-4x}{6} + \frac{1+x}{3}$

(16.) $\frac{1}{3x-4} = \frac{2}{3x+7}$.

(17.) $\frac{3}{x} + 1 = \frac{x}{x-2}$.

(18.) $\frac{7}{7x-1} + \frac{6}{6x-1} = \frac{10}{5x-1}$.

(19.) $\frac{x+2\frac{3}{4}}{5} - \frac{1}{3}\left\{5x + 4 - \frac{1}{4}\left(2 - \frac{x-5}{4}\right)\right\} + \frac{11x+1}{6} =$

(20.) $\frac{2x+4}{x-1} - 1 = \frac{x+9}{x}$.

(21.) $\frac{2}{2x^2-3x-2} = \frac{1}{x^2-4x+9}$.

(22.) $\frac{8x-3}{2} + 4 = \frac{x^2+5}{x+5} + 3x$.

(23.) $\frac{3-2x}{x-1} = \frac{5}{7(x-3)} - 2$.

(24.) $\frac{x-6}{2x-5} \cdot \frac{3x-8}{x-5} = \frac{3}{2}$.

(25.) $\frac{(x-2)(2x+3)(3x-2)}{(3x+1)(2x-1)(x-1)} = 1$.

(26.) $\frac{x-3}{x-5} - \frac{x-6}{x-8} = \frac{x-2}{x-4} - \frac{x-5}{x-7}$.

(27.) $\dfrac{1+x}{4+x} - \dfrac{3+x}{6+x} = \dfrac{2+x}{5+x} - \dfrac{4+x}{7+x}$.

(28.) $\dfrac{x}{1-2x} + \dfrac{2-3x^2}{2x^2-\frac{1}{2}} + 2 = 0$.

(29.) $\dfrac{x}{a} + \dfrac{x}{b} = \dfrac{1}{a^2b} + \dfrac{1}{ab^2}$.

(30.) $\dfrac{a}{x-a} - \dfrac{b}{x-b} = \dfrac{a-b}{x}$.

(31.) $\dfrac{a^2-x}{a-1} + \dfrac{b^2-x}{b+1} = a + b$.

(32.) $\dfrac{x}{m^2-1+nx} = \dfrac{m-x}{mn+1-nx}$.

(33.) $\dfrac{3x^2-4x+2}{2x^2-x-2} = \dfrac{3x^2-4x+11}{2x^2-x+4}$.

(34.) $\dfrac{4x+(a+1)^2}{4x-(a+1)^2} = \dfrac{x+a-1}{x-a-1}$.

(35.) $\dfrac{a}{x-b} + \dfrac{b}{x-a} = \dfrac{a+b}{x-a-b}$.

(36.) $\left(1+\dfrac{1}{a}\right)\left(1-\dfrac{1}{x}\right) - \left(1-\dfrac{1}{b}\right)\left(1+\dfrac{1}{x}\right) + \dfrac{2}{x} = \dfrac{1}{c}$.

1. Problems involving Fractions.

) A draper bought a piece of cloth at 2s. per yard. old one-third of it at 2s. 6d. per yard, one-fourth at d., and the remainder at 2s. 1d., gaining 12s. 8d. many yards did the piece contain?

et x = the number of yards.
hen $24x$ = the cost in pence.

nd $\dfrac{x}{3} \times 30 + \dfrac{x}{4} \times 27 + \dfrac{5x}{12} \times 25$ = total selling price.

But by the question, this equals $24x+152$.

$$\therefore\ 10x+\frac{27}{4}x+\frac{125}{12}x = 24x+152.$$

Cl. off fractions, $120x+81x+125x = 288x+1824$.

Collecting, $326x-288x = 1824$.

$$\therefore\ 38x = 1824, \text{ and } x = 48.$$

(2.) One pipe can fill a cistern in a hours; another ca do it in b hours; in what time could the two runnir together fill it?

And if a third pipe could empty the cistern in c hour how long would it take to do this if the first two we running at the same time?

As the first pipe fills the cistern in a hours, it will f the $\frac{1}{a}$th part of it in one hour; similarly the second w do the $\frac{1}{b}$th part in an hour; and the two together will $\frac{1}{a}+\frac{1}{b}$.

Now, if x = the number of hours which the two ning together will require to fill the cistern, $\frac{1}{x}$ will sent what they will do in one hour, and therefore

$$\frac{1}{a}+\frac{1}{b}=\frac{1}{x},$$

from which $(a+b)x=ab$, and $x=\frac{ab}{a+b}$

Again, $\frac{1}{c}$ will stand for the part of the cistern be emptied in one hour by the third pipe, but flow in, in the same time, the part actually em hour will be $\frac{1}{c}-\left(\frac{1}{a}+\frac{1}{b}\right)$, which may be put

y be the number of hours required to empty the cistern when the three pipes are all running together.

$$\frac{1}{c} - \left(\frac{1}{a} + \frac{1}{b}\right) = \frac{1}{y}.$$

$$(ab - bc - ac)y = abc, \text{ and } \therefore\ y = \frac{abc}{ab - bc - ac}.$$

(3.) A person can row 12 miles up a stream and back again in 8 hours, and he finds that in equal times he goes down half as far again as he does up. Find his time of going and returning, his rate of rowing, and the rate of the stream.

Let x = the number of miles he moves per hour going up.

$\frac{3}{2}x$ = the number of miles he moves per hour going down.

Then $\frac{12}{x}$ = the number of hours to go up.

$\frac{8}{x}$ = the number of hours to go down.

$$\therefore\ \frac{12}{x} + \frac{8}{x} = 8.$$

From which $8x = 20$;

Or, $x = 2\frac{1}{2}$, and $\frac{3}{2}x = 3\frac{3}{4}$.

$$\frac{12}{x} = \frac{12}{2\frac{1}{2}} = 4\frac{4}{5} = \text{number of hours to go up.}$$

$$\frac{8}{x} = \frac{8}{2\frac{1}{2}} = 3\frac{1}{5} = \text{number of hours to go down.}$$

The difference between his rate per hour going and his rate returning must be twice the rate of the stream; for in the one case it retards and in the other assists him; and therefore

$$\frac{1}{2}(3\tfrac{3}{4} - 2\tfrac{1}{2}) = \frac{1}{2} \times 1\tfrac{1}{4} = \frac{5}{8} = \text{rate of stream per hour.}$$

This, added to $2\frac{1}{2}$, or taken from $3\frac{3}{4}$, must give his rate of rowing, which is therefore

$$= 2\tfrac{1}{2} + \frac{5}{8} = 3\tfrac{3}{4} - \frac{5}{8} = 3\tfrac{1}{8}.$$

(4.) The difference between two numbers is 8, and their ratio is $\frac{3}{5}$; what are the numbers?

Let x = the first.
Then $x+8$ = the second.
And $\frac{x}{x+8} = \frac{3}{5}$.
From which $5x = 3x+24$.
$\therefore x=12$, and $x+8=20$.

(5.) A railway train goes from A to B in 4 hours. On its return journey it does three-fifths of the way at a speed increased by six miles an hour, but is afterwards compelled to reduce this rate by 12 miles. It nevertheless does the whole distance in the same time it took to go. Find how far it is from A to B.

Let x = the required distance in miles.

Then $\frac{x}{4}$ = rate per hour going.

$\frac{x}{4}+6$ and $\frac{x}{4}-6$ = rates returning.

As the distance travelled, divided by the rate per hour, gives the time, we have by the question—

$$\frac{\frac{3}{5}x}{\frac{x}{4}+6} + \frac{\frac{2}{5}x}{\frac{x}{4}-6} = 4,$$

$$\frac{3}{5}\cdot\frac{x^2}{4} - \frac{18}{5}x + \frac{2}{5}\cdot\frac{x^2}{4} + \frac{12}{5}x = 4\left(\frac{x^2}{16} - 36\right)$$

$$\frac{x^2}{4} - \frac{6}{5}x = \frac{x^2}{4} - 144.$$

$$\therefore \frac{6}{5}x = 144, \text{ and } x = 120.$$

EXAMPLES FOR PRACTICE.—XI.

(1.) Find the number whose fourth, fifth, and eighth parts added together make up $11\frac{1}{2}$.

(2.) A cistern is $\frac{5}{6}$ths full; after 13 gallons are drawn off, it is found to be still $\frac{3}{4}$ths full. How much can it contain?

(3.) Divide 93 into two parts, such that $\frac{3}{4}$ths of the one shall equal $\frac{4}{5}$ths of the other.

(4.) Two pipes running together can fill a cistern in 6 hours; one alone would do it in 10 hours. How long would the other take?

(5.) In a factory where 17 men, 25 women, and 11 children are employed, the sum required to pay the weekly wages is £38, 4s. 9d. If a woman receive two-thirds of a man's wage, and a child three-eighths of a woman's, what is paid to each?

(6.) A person playing at cards first lost one-sixth of his money, and then gained 10s.; he next lost one-fifth of what he had, and then gained 8s.; afterwards he lost one-fourth of the sum he then had, and regained 9s. On counting his gains and losses for the whole evening, he found them equal. What money had he?

(7.) A cruiser observed a vessel 12 miles distant, and immediately gave chase, the rate of pursuer to that of pursued being as 12 to 11. At the end of two hours the wind freshened, whereby the speed of the former was increased by one-half and that of the latter by four-elevenths of its previous rate. The capture was made in 9 hours and 20 minutes. What was the speed of the two vessels at starting?

(8.) A lady, making some purchases, spent at the first place one shilling more than the fourth part of her money; at the second, one shilling more than the third part of the remainder; and at the third, one shilling more than the half of what still was left. She took home twopence. What did she take out?

(9.) A and B enter into partnership. A puts in £1000 more than B, but at the end of three months he withdraws £2000; at the end of ten months B withdraws £240; and when the year is closed, it is found that A's profits are to B's as 5 is to 6. Find the capital of each.

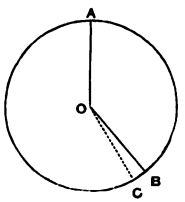
(10.) There are four numbers, the first, second, and third of which are respectively 11, 7, and 3 greater than the fourth; and the ratio of the first to the second is equal to that of the third to the fourth. Find the numbers.

(11.) A and B can together do a piece of work in 7 days. A, with the help of C for 3 days, could finish it in 8 days; and B, with C's help for 2 days, could do it in 10 days. How long would each take to do it separately?

Put x = number of days C takes.

(12.) A steam vessel usually occupied 6 days in going between the ports M and N. On one occasion, after it had been a day out, an accident caused the speed to be reduced 40 miles a day. Having gone 200 miles at this rate, and repaired its machinery, it was able to increase its speed to 10 miles a day more than its average, and thus finish its voyage within the usual time. Required the distance between M and N.

Questions regarding the relative positions of the hands of a clock or watch are frequently set. The following examples will show how they may be wrought:—



(*a*) At what time between 5 and 6 will the hands of a clock coincide?

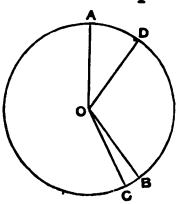
In the accompanying figure, let OA represent the position of the minute hand at each hour, OB the position of the hour hand at 5 o'clock, and OC the position of the two when they are together. Now the minute hand travels twelve times as fast as the hour hand, for it completes the circle while

the other passes through one hour. If then x = the number of minute spaces in AC, $\frac{x}{12}$ will equal the number in BC; and there are 25 in AB. We have therefore—

$$x = \frac{x}{12} + 25.$$

From which $x = 27\frac{3}{11}$ = the number of minutes after 5 when the hands coincide.

(*b*) When will the hands of a clock be 20 minutes apart between the hours of 5 and 6?



At 5 o'clock the hands are 25 minutes apart, and as they draw together and then separate, there must be two positions in which the space between them amounts to 20 minutes. Let OC, OD be the first of these positions.

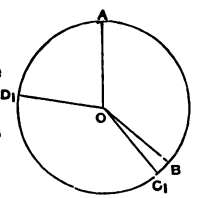
If x = number of minute spaces in AD,

Then $\frac{x}{12}$ = number of minute spaces in BC;

while DC contains 20, and AB 25, by the question.

This gives $x + 20 = AC = 25 + \frac{x}{12}$.

From which $x = 5\frac{5}{11}$ min.



Let OC_1, OD_1 be the second of the positions.

If x_1 = the number of minute spaces in ABD_1, then $x_1 - 20 = AC_1 = \frac{x_1}{12} + 25$.

From which $x_1 = 49\frac{1}{11}$ min.

Generally, if m be the hour after which the hands are to be n minutes apart, and y the number of minutes after m when this shall happen, we have—

$$y \pm n = \frac{y}{12} + 5m$$

$$\text{And } \therefore y = \frac{12}{11}(5m \mp n).$$

(13.) At what time will the hands of a clock be together between 6 and 7 o'clock?

(14.) When will the hands of a clock be at right angles to one another between the hours of 7 and 8?

(15.) The hands of a watch are in the same straight line at 6 o'clock. When will they next be in a like condition?

(16.) How often will the hands of a clock be 25 minutes apart between the hours of 4 and 6?

(17.) A person looking at his watch between 10 and 11 mistook the hands, and imagined the time to be 54 minutes earlier than it really was. What time was it?

(18.) A asks B what time it is, and receives for answer that in 4 minutes more the minute hand will be as much past 3 as the hour hand is now past 2. What o'clock is it?

From Wharton's Examples in Algebra.

(19.) A person pays an income-tax of 7d. in the £, and a poor-rate exceeding it by £22, 10s., and has £486 left. Find his income.

(20.) In the election of a Member of Parliament, $\frac{1}{16}$ of the constituency refuse to vote; and of two candidates the one who is supported by $\frac{19}{40}$ of the whole constituency is returned by a majority of 5. Find the number for each candidate.

(21.) Says A to B, If you give me 10 guineas of your money, I shall then have twice as many as you have left. But says B to A, Give me 10 of your guineas, and then I shall have three times as many as you. How many had each?

(22.) A person sets out from A and travels towards B at the rate of $3\frac{1}{2}$ miles an hour; 40 minutes afterwards another sets out from B to meet him, travelling at the rate of $4\frac{1}{2}$ miles an hour, and he goes half a mile beyond the

middle of the distance before he meets the first traveller. Find the distance between A and B.

(23.) A and B start to run a race to a certain post and back again. A, returning, meets B 90 yards from the post, and arrives at the starting-place 3 minutes before him. If he had then returned immediately to meet B a second time, he would have met him at one-sixth of the distance between the post and starting-place. Find the length of the course, and the duration of the race.

(24.) A entered into a canal speculation with 14 others, and the profits in this concern amounted in all to £595 more than five times the price of an original share. Seven of his former partners joined him in a scheme for navigating the canal with steamboats, each venturing a sum less than his former gains by £173. But the steamboats blowing up, A found he had lost £419 by them; for the company not only never recovered the money advanced, but lost all they had gained by digging the canal, and £368 besides. What were the prices of shares in the two concerns originally?

CHAPTER IX.

SIMULTANEOUS EQUATIONS.

112. Hitherto our equations have each contained only one unknown quantity; but equations may be formed in which there are two, three, or more quantities whose values are unknown.

Let us first look at one containing two such quantities, say $7x - 3y = 29$.

As we do not yet know any method of solving such an expression, we shall make a guess at the value of one of the quantities, and from that deduce the other.

Suppose $y = 0$, then $7x = 29$, and $x = 4\frac{1}{7}$.
Suppose $y = 1$, then $7x = 32$, and $x = 4\frac{4}{7}$.
Suppose $y = 2$, then $7x = 35$, and $x = 5$.
Suppose $y = 3$, then $7x = 38$, and $x = 5\frac{3}{7}$.
. . . . . .

It is easily seen that for every value of y there will be a corresponding value of x; and consequently the number of solutions of the above equation is quite unlimited.

Such an equation is said to be indeterminate.

If we take now another equation, say $3x + 2y = 19$, containing the same two unknown quantities, we shall get from it also an infinite number of solutions:—

If $y = 0$, then $3x = 19$, and $x = 6\frac{1}{3}$.
If $y = 1$, then $3x = 17$, and $x = 5\frac{2}{3}$.
If $y = 2$, then $3x = 15$, and $x = 5$.
If $y = 3$, then $3x = 13$, and $x = 4\frac{1}{3}$.
. . . . . .

ice that in all the values of x and y that are found
or that might be found, there is only one pair the
n both equations—namely, $x = 5$, when $y = 2$.
ıppears from this that when these equations are
separately they afford an infinite number of solu-
but when taken together they yield only one.
ı equations are called simultaneous.
'e had had the two equations

$$7x - 3y = 29 \text{ and } 3x + 2y = 19$$

ıs for solution, we should have endeavoured to find
ly those values of x and y that satisfy both equa-

ırder to do this, it is necessary to combine the two
ɔns in such a manner as to make one of the un-
. quantities disappear, and so form one equation
ne unknown quantity.
this purpose several methods are employed.

First Method.

We may find from one of the equations an expres-
ır either of the unknown quantities in terms of the
and the known quantities, and substitute this in the
equation.

Illustrative Example.

ing the above equations, we have, from the first,
$\frac{y+29}{7}$; and putting this in the second instead of x,
tain $3\left(\frac{3y+29}{7}\right) + 2y = 19,$

an equation with only one unknown quantity, which, being solved, gives—

$$9y + 87 + 14y = 133, \text{ or } y = 2, \text{ and}$$

$$x = \frac{3y+29}{7} = \frac{6+29}{7} = \frac{35}{7} = 5,$$

the same values as before.

This method of solution is called "Substitution."

Second Method.

114. We may find from *each* of the equations an expression for *one* of the unknown quantities, as in the first method; and by equating these expressions, form an equation containing only one unknown quantity.

Illustrative Example.

Given $5x - 2y = 1$ and $2x + 3y = 27$, to find x and y.

From first, $y = \frac{5x-1}{2}$; from second, $y = \frac{27-2x}{3}$

As $\frac{5x-1}{2}$ and $\frac{27-2x}{3}$ are each equal to y, they are equal to one another.

$$\therefore \frac{5x-1}{2} = \frac{27-2x}{3}$$

This being solved, gives $x = 3$, and from either of the expressions for y we may find its value. Say from the first, then

$$y = \frac{5x-1}{2} = \frac{15-1}{2} = \frac{14}{2} = 7.$$

This method of solution is called "Equating" or "Comparison."

Third Method.

115. We may eliminate one of the unknown quantities by rendering its coefficient the same in both equations and

then subtracting the one equation from the other, or adding the two together, according as the signs of the quantities to be got rid of are like or unlike.

Illustrative Examples.

(1.) Given $9x - 4y = 38$ and $6x + 5y = 56$, to find x and y.

Suppose we wish to find y first, we shall then equalize the coefficients of x by bringing them to their least common multiple, 18. To do this, the 9 must be multiplied by 2, and the 6 by 3.

But if only one term of an equation be multiplied, the value of the unknown quantity will plainly be altered; whilst if every term be multiplied, it will remain unchanged.

Multiply, then, every term of the first equation by 2, and every term of the second by 3.

Note.—The different processes employed in the solution of an equation may be indicated at the side; the line operated on being shown by a figure enclosed in brackets.

$$\begin{array}{lll} [1] \times 2, & 18x - 8y = 76 & [3] \\ [2] \times 3, & 18x + 15y = 168 & [4] \end{array}$$

As the $18x$ is positive in both equations, it will disappear by the subtraction of the one line from the other.

In performing this, it is a matter of some convenience to subtract that line which will leave the remaining unknown quantity with a positive sign.

Taking, then, the upper from the under line, we have

$$[4] - [3], \qquad 23y = 92, \text{ or } y = 4.$$

We may now find the value of x by substituting this

value of y in one of the given equations, say the second, then—

$$x = \frac{1}{6}(56 - 5y) = \frac{1}{6}(56 - 20) = \frac{36}{6} = 6.$$

Or we may find x in the same manner as we have done y, by making the coefficients of y the same in both equations —this latter method being generally preferable, as it yields two independent solutions.

The L.C.M. of 4 and 5 being 20, we multiply the first equation by 5 and the second by 4; this gives—

$$45x - 20y = 190 \text{ and}$$
$$24x + 20y = 224.$$

By adding the two lines together, the $20y$ will disappear, and there results—

$$69x = 414, \text{ or } x = 6.$$

The required values are therefore $x = 6$, $y = 4$.

This method of solution is known as that of "Equalizing Coefficients." It is in more general use than either of the two previous methods.

The following examples will afford additional illustration of its application :—

(2.) Given $\left\{\begin{array}{ll} 3x - 7y = -9 & [1] \\ 5y - 2x = 8 & [2] \end{array}\right\}$ to find x and y.

In equalizing the coefficients, it will be convenient to place x under x and y under y.

To find y first :

[1] × 2,	$6x - 14y = -18$	[3]
[2] × 3,	$-6x + 15y = 24$	[4]
[3] + [4],	$\therefore y = 6.$	

To find x :

[1] × 5,	$15x - 35y = -45$	[5]
[2] × 7,	$-14x + 35y = 56$	[6]
[5] + [6],	$\therefore x = 11.$	

3.) Given $\left\{\begin{array}{ll} ax+by=c & [1] \\ mx-ny=p & [2] \end{array}\right\}$ to find x and y.

$[1] \times n, \quad anx+bny=cn \quad [3]$

$[2] \times b, \quad bmx-bny=bp \quad [4]$

$[3]+[4], \quad (an+bm)x=cn+bp$

$$\therefore x=\frac{cn+bp}{an+bm}.$$

$[1] \times m, \quad amx+bmy=cm \quad [5]$

$[2] \times a, \quad amx-any=ap \quad [6]$

$[5]-[6], \quad (an+bm)y=cm-ap$

$$\therefore y=\frac{cm-ap}{an+bm}.$$

4.) Given $\left\{\begin{array}{ll} \frac{8}{x}+\frac{9}{y}=5 & [1] \\ \frac{16}{x}-\frac{3}{y}=3 & [2] \end{array}\right\}$ to find x and y.

Here solve for $\frac{1}{x}$:

$[1], \quad \frac{8}{x}+\frac{9}{y}=5 \quad [3]$

$[2] \times 3, \quad \frac{48}{x}-\frac{9}{y}=9 \quad [4]$

$[3]+[4], \quad \frac{56}{x}=14,$

Or, $\frac{1}{x}=\frac{14}{56}=\frac{1}{4}$, and $\therefore x=4$.

Now solve for $\frac{1}{y}$:

$[1] \times 2, \quad \frac{16}{x}+\frac{18}{y}=10 \quad [5]$

$[2], \quad \frac{16}{x}-\frac{3}{y}=3 \quad [6]$

$[5]-[6], \quad \frac{21}{y}=7,$

$$\text{Or, } \frac{1}{y} = \frac{7}{21} = \frac{1}{3}, \text{ and } \therefore y = 3.$$

(5.) Given $\frac{5xy}{3x+4y} = 6$, and $\frac{xy}{2x-3y} = 12$, to find x and y.

From [1], $$\frac{xy}{3x+4y} = \frac{6}{5}.$$

Now invert both sides of the equation. This gives—

$$\frac{3x+4y}{xy} = \frac{5}{6}, \text{ or } \frac{3}{y} + \frac{4}{x} = \frac{5}{6} \quad [3]$$

And similarly from [2],

$$\frac{2x-3y}{xy} = \frac{1}{12}, \text{ or } \frac{2}{y} - \frac{3}{x} = \frac{1}{12} \quad [4],$$

a pair of equations from which x and y may be found in last example.

$$[3] \times 2, \qquad \frac{6}{y} + \frac{8}{x} = \frac{5}{3} \quad [5]$$

$$[4] \times 3, \qquad \frac{6}{y} - \frac{9}{x} = \frac{1}{4} \quad [6]$$

$$[5] - [6], \qquad \frac{17}{x} = \frac{5}{3} - \frac{1}{4} = \frac{20-3}{12} = \frac{17}{12}$$

$$\therefore \frac{1}{x} = \frac{1}{12}, \text{ or } x = 12.$$

$$[3] \times 3, \qquad \frac{9}{y} + \frac{12}{x} = \frac{5}{2} \quad [7]$$

$$[4] \times 4, \qquad \frac{8}{y} - \frac{12}{x} = \frac{1}{3} \quad [8]$$

$$[7] + [8], \qquad \frac{17}{y} = \frac{5}{2} + \frac{1}{3} = \frac{15+2}{6} = \frac{17}{6}.$$

$$\therefore \frac{1}{y} = \frac{1}{6}, \text{ or } y = 6.$$

(6.) Given $\left.\begin{cases} \frac{x}{a}+\frac{y}{b}=m-\frac{x}{c} & [1] \\ \frac{x}{b}+\frac{y}{a}=n-\frac{y}{c} & [2] \end{cases}\right\}$ to find x and y.

Clear away fractions and transpose.

Then $(a+c)bx+acy=abcm$ [3]

And $acx+(a+c)by=abcn$ [4]

[3] $\times (a+c)b$, $(a+c)^2b^2x+abc(a+c)y=ab^2c(a+c)m$

[4] $\times ac$, $a^2c^2x+abc(a+c)y=a^2bc^2n$.

$\therefore \{(a+c)^2b^2-a^2c^2\}x=abc\{b(a+c)m-acn\}$

And $x=\frac{abc\{b(a+c)m-acn\}}{b^2(a+c)^2-a^2c^2}$

[3] $\times ac$, $abc(a+c)x+a^2c^2y=a^2bc^2m$

[4] $\times b(a+c)$, $abc(a+c)x+b^2(a+c)^2y=ab^2c(a+c)n$.

$\therefore \{b^2(a+c)^2-a^2c^2\}y=abc\{b(a+c)n-acm\}$

And $y=\frac{abc\{b(a+c)n-acm\}}{b^2(a+c)^2-a^2c^2}$.

EXAMPLES FOR PRACTICE.—XLI.

Find the values of x and y in the following equations:—

(1.) $x+y=7$, $x-y=3$.

(2.) $4x+y=17$, $3x+5y=17$.

(3.) $2x-5y=4$, $4x-7y=14$.

(4.) $\frac{3}{4}x-y=4$, $6x-13y=7$.

(5.) $\frac{1}{2}x+\frac{1}{3}y=7$, $4x-3y=5$.

(6.) $\frac{3}{5}x+2y=18$, $2x-\frac{1}{11}y=18$.

(7.) $4x-9=13-5y$,
$x-12=8y+12$.

(8.) $\frac{5x-6}{4}+3y=30$,

$3x+5=\frac{5y}{4}+2y-3$.

(9.) $x+\frac{2y-1}{3}=y+\frac{3}{4}(x+1)$,
$5x-7y=19$.

(10.) $\frac{3x-5}{4}+6y=3x-\frac{13y+1}{5}$,
$2x-\frac{10y-3}{3}=\frac{17y+3}{9}+\frac{x+y}{2}$

(11.) $\cdot5x+\cdot08y=7$, $\cdot3y-\frac{x-\cdot2y}{\cdot8}=20$.

(12.) $\cdot4x-\frac{\cdot05y-\cdot1}{\cdot15}=\frac{\cdot6x+2\cdot25y}{5}$,
$\cdot3x+\frac{4y-5}{9}=11-\cdot2(x-10)$.

(13.) $\frac{9}{x}+\frac{4}{y}=5$, $\frac{15}{x}-\frac{8}{y}=1$.

(14.) $(x+2)(y-5)=xy$, $\frac{3}{4}x+\frac{2}{5}y=5\frac{1}{2}$.

(15.) $9x+20y=xy$, $25y-6x=7xy$.

(16.) $(x+3)(y-3)=(x-3)y$,
$(x+2)(y-1)=(x-1)(y+1)$.

(17.) $\frac{3x+5}{8}-\frac{16}{y}=6$, $2xy=3(5y-8)$.

(18.) $\frac{6}{x+y}+\frac{5}{x-y}=7$, $\frac{15}{x+y}-\frac{2}{x-y}=3$.

(19.) $5x-\frac{21}{y-1}=18$, $(x-7)y=x-13$.

(20.) $(a-b)x+(a+b)y=a$,
$(a+b)x-(a-b)y=b$.

(21.) $abx+bcy=acxy$, $bcx+acy=abxy$.

(22.) $(a-b+c)x-(a+b-c)y=\dfrac{2ac}{a-b}$,

$\dfrac{x+y}{x-y}=\dfrac{a}{b}$.

(23.) $\dfrac{cxy}{x+y}=\dfrac{a^2b^2}{a^3+b^3}$; $\dfrac{ax+by}{xy}=\dfrac{(a^2+b^2)c}{ab}$.

(24.) $\dfrac{b^2}{x}+\dfrac{a^2}{y}=a^2-b^2$,

$\dfrac{a^2}{x}+\dfrac{b^2}{y}=(a^2-b^2)(a^2-1+b^2)$.

i. Problems producing equations with two unknown tities.

) Find two numbers such that two-thirds of the first three-fifths of the second amount to 14, while five-hs of the first and three-fourths of the second come ;.

Let x = the first, and y = the second.

Then by the question, $\frac{2}{3}x+\frac{3}{5}y = 14$ [1]

And $\frac{5}{8}x+\frac{3}{4}y = 15$ [2]

rst clear off fractions:

[1] × 15, $10x+9y = 210$ [3]

[2] × 8, $5x+6y = 120$ [4]

ualize the coefficients of x:

[4] × 2, $10x+12y = 240$ [5]

[5] − [3], $3y = 30$ $\therefore\ y=10.$

ualize the coefficients of y:

[3] × 2, $20x+18y = 420$ [6]

[4] × 3, $15x+18y = 360$ [7]

[6] − [7], $5x=60$ $\therefore\ x=12.$

Proof, $\frac{2}{3}$ of $12+\frac{3}{5}$ of $10=8+6=14$.

And $\frac{5}{8}$ of $12+\frac{3}{4}$ of $10=7\frac{1}{2}+7\frac{1}{2}=15$.

(2.) A person going from A to B, a distance of 33 miles, walked 12 miles and rowed the rest, doing the whole in 10 hours. Coming back, he rowed 14 miles and walked the rest; and when still one mile from home he found that he had been as long on the return journey as it had previously taken him to go the whole way. Required his rates of walking and rowing.

Let x = the number of miles he walked per hour.

y = the number of miles he rowed per hour.

Then $\frac{12}{x}$ = the number of hours he took to walk 12 miles.

$\frac{21}{y}$ = the number of hours he took to row 21 miles.

$\therefore \quad \frac{12}{x} + \frac{21}{y} = 10$, time in going 33 miles.

Similarly $\frac{18}{x} + \frac{14}{y} = 10$, time in returning 32 miles.

Equalize the coefficients of $\frac{1}{y}$:

$$\frac{24}{x} + \frac{42}{y} = 20, \text{ and } \frac{54}{x} + \frac{42}{y} = 30,$$

$$\therefore \frac{30}{x} = 10.$$

And $x = 3$ = number of miles he walked per hour. By substitution, or by equalizing the coefficients of x, we will find $y = 3\frac{1}{2}$ = number of miles he rowed per hour.

(3.) A certain number of two places is five times the sum of its digits; and if 9 be added to it, the one digit will take the place of the other. Find the number.

Let x = the digit in the ten's place,

And y = the digit in the unit's place.

Then $10x+y$ = the number,

And $10y+x$ = the number inverted.

By the question, $10x+y=5(x+y)$ [1]

And $10x+y+9=10y+x$ [2]

From [1], $5x=4y$, and $x=\frac{4}{5}y$ [3]

From [2], $9x+9=9y$, and $x=y-1$ [4]

[3] = [4], $\therefore\ y-1=\frac{4}{5}y$, and $y=5$.

Also $x=\frac{4}{5}y=4$.

The number, therefore, is 45.

4.) If 3 be added to both numerator and denominator a certain fraction, it will become $\frac{3}{4}$; but if 5 be subcted from each, it will equal $\frac{1}{2}$. Find the fraction.

Let x = the numerator, y = the denominator.

Then $\frac{x+3}{y+3}=\frac{3}{4}$, and $\frac{x-5}{y-5}=\frac{1}{2}$.

From [1], $4x+12=3y+9$, or $4x-3y=-3$.

From [2], $2x-10=y-5$, or $2x-y=5$.

Solving in usual way gives $x=9$, $y=13$; and the ction is $\frac{9}{13}$.

Performing the additions and subtractions indicated, we e

$$\frac{9+3}{13+3}=\frac{12}{16}=\frac{3}{4}, \text{ and } \frac{9-5}{13-5}=\frac{4}{8}=\frac{1}{2}.$$

PROBLEMS FOR SOLUTION.—XLII.

Note.—Some of the following problems may be solved by using only one unknown quantity, and many of those already set may be more easily solved by two.

1.) If A's money were increased by £10, he would e three times as much as B; while if it were diminished £10, he would have only twice as much. What has h?

2.) If one be taken from the numerator and added to denominator of a certain fraction, it will become $\frac{3}{4}$;

but if five be thus subtracted and added, the value will be $\frac{1}{2}$. Find the fraction.

(3.) A person having a sum of £10 to spend can buy with it either x yards at 12s. and y yards at 4s., or $x+1$ yards at 13s. and $y-1$ yards at 3s. If he buy the former, how many yards of each kind does he get?

(4.) A traveller sets out from M for N, a distance of 75 miles, at the same time that two others leave N to go to M. When he has walked $40\frac{1}{2}$ miles he meets one of the others, and $4\frac{1}{2}$ miles further on, the second, who has fallen five-sixths of a mile per hour behind his neighbour. Required the rate per hour at which each walks.

(5.) A purse, whose total value is £33, 12s., contains 41 coins, made up of guineas, sovereigns, and crowns. The number of guineas is greater than the number of sovereigns by twice the excess of the number of sovereigns over the number of crowns. How many pieces of each kind were in it?

(6.) A, B, and C have a pounds among them. A and C together have m times what B has, while B and C between them have n times what A has. How much has each got?

(7.) The expenses of a dredging expedition were equally paid by the company present. If there had been 5 persons fewer, each would have paid 1s. 6d. more; and if 5 persons more, the charge to each would have been 1s. less. How many were present, and what did the expedition cost?

(8.) A waterman, after rowing 5 miles up a river, increased his speed by 1 mile per hour, and reached his destination 3 hours thereafter. On his return, he rowed 6 miles at his first speed, and then decreasing it by 1 mile per hour reached home in another hour. If, in going up, he had delayed increasing his speed for 1 hour, he would have been half an hour later in arriving at the end of his

urney. What was his rate of rowing, and the rate of e stream per hour?

(9.) A number of three digits has 5 in the unit's place, d the middle figure is half the sum of the other two. 108 be added to the number, the hundred's figure will ke the place of the unit's, and the unit's the place of the n's. Find the number.

(10.) Two burghs unite in sending a member to Parlia- ent. At a contested election the successful candidate, A, d a majority of 160 in the first burgh, and a minority of) in the second. If one-fortieth of his supporters in the st place, and one-twentieth of those in the second, had oted for B, his opponent, A would still have had a ajority of 2 over the whole. But if a third candidate, , had come in and taken away one-fourth of A's voters nd one-sixth of B's, B would have been elected by a ajority of 25 over A. How many voted for A and B in ach of the burghs?

(11.) A botanical society went on an excursion in search f specimens, and for convenience broke up into two arties. On the way out, the first party obtained 18 nore specimens than the second, the number gathered by ach member of the first company being on an average 5 reater than the number of individuals in the second; hile the number gathered by each of the second company as 2 greater than the number of persons in the first. n returning, each of the first party got 7 specimens fewer han the number of persons in the second, and each of the econd got 5 fewer than the number of persons compos- ng the first; and now the second company had on the hole one more than the first. How many went on the xcursion?

(12.) Two men, David and George, with their wives, Mary and Jane, went to make purchases. Each man

spent 4s. less than his wife, but David spent 10s. more than his friend's wife. Of the whole expenditure, George's share was one-tenth, and Mary's one-sixth. Which of the men was Jane's husband?

Selected Examples.

(13.) Find two numbers such that when the greater is divided by the less the quotient is 4 and the remainder 3; and when the sum of the numbers is increased by 38, and the result divided by the greater of the two numbers, the quotient is 2 and the remainder 2.

(14.) A man invested 2s. 6d. in apples and pears, buying the apples at 4 a penny and the pears at 5 a penny. He sold half his apples and one-third of his pears for 13d., which was at the rate at which he bought them. How many did he buy of each sort?

(15.) There is a certain rectangular floor, such that if it had been 2 feet broader and 3 feet longer it would have been 64 square feet larger; but if it had been 3 feet broader and 2 feet longer, it would have been 68 square feet larger. Find the length and breadth of the floor. (*The length multiplied by the breadth gives the area.*)

(16.) A and B ran a race which lasted 5 minutes. B had a start of 20 yards; but A ran 3 yards while B was running 2, and won by 30 yards. Find the length of the course and the speed of each.

(17.) A hare is 40 of her own leaps before a greyhound, and takes 5 leaps for the greyhound's 4; but 3 of the greyhound's leaps are equal to 4 of the hare's. How many leaps must the greyhound take to catch the hare?

Let x = number of leaps taken by greyhound,

Then $\frac{5}{4}x$ = number of leaps taken by hare in same time.

Let y = length in feet of a greyhound's leap,

Then $\frac{3}{4}y$ = length in feet of a hare's leap.

The distance run by the greyhound before making the capture will be xy.

The distance run by the hare before being captured will be $\frac{5}{4}x \times \frac{3}{4}y = \frac{15}{16}xy$.

And the distance of the hare in front of the greyhound at the beginning of the pursuit will be $40 \times \frac{3}{4}y = 30y$.

The equation, therefore, is—

$$xy = \tfrac{15}{16}xy + 30y.$$

Divide out y, then $x = \frac{15}{16}x + 30$.

From which, $x = 480$ = number of greyhound's leaps.

And $\frac{5}{4}x = 600$ = number of hare's leaps.

Questions of this character may be solved by using one unknown quantity only, but the help of a second is useful in rendering the explanation of the solution somewhat plainer.

(18.) A greyhound starts in pursuit of a hare at the distance of 50 of his own leaps from her. He makes 3 leaps while the hare makes 4, and he covers as much ground in 2 leaps as the hare does in 3. How many leaps does each make before the hare is caught?

(19.) Two trains, 92 feet long and 84 feet long respectively, are moving with uniform velocities on parallel rails in opposite directions, and are observed to pass each other in one second and a half; but when they are moving in the same direction, their velocities being the same as before, the faster train is observed to pass the other in 6 seconds. Find the rate in miles per hour at which each train moves.

(20.) A boy at a fair spends his money in oranges. If he had received 5 more for his money, they would have averaged a halfpenny each less; if 3 less, a halfpenny each more. How much did he spend?

(21.) A person travelled a journey at a certain rate.

Had he travelled half a mile an hour faster, he would have performed the journey in four-fifths of the time; but had he travelled half a mile an hour slower, he would have been $2\frac{1}{2}$ hours longer on the road. Find the distance, and his rate of travelling.

(22.) A stage-coach carries 6 inside; the fare outside is 13s., and one-third of the sum of the outside fares exceeds one-sixth of those inside by £1, 5s. 4d. An opposition arising, the coachman loses 3 outside and 2 inside passengers, and also reduces the inside fare by 5s., and halves the outside, and then the whole loss is £7, 0s. 6d. Find the number of outside places, and the inside fare.

(23.) A farmer's rent was £50 a year, and his annual expenditure (including the assessed taxes, which amounted to $\frac{1}{6}$ of his expenses) was such that he was able to pay his landlord only £30. The year following, his rent was lowered 20 per cent., the taxes were reduced one-half, and agricultural produce increased in value one-third. In consequence, he was enabled to pay his rent and former debt, and to lay by £5. What was his expenditure, and the value of his produce each year?

(24.) A person has £27, 6s. in guineas and crown-pieces out of which he pays a debt of £14, 17s., and finds he has exactly as many guineas left as he has paid away crowns, and as many crowns as he has paid away guineas. How many of each had he at first?

117. Equations with Three Unknown Quantities.—When one equation containing two unknown quantities is given for solution, we have seen (Art. 112) that an infinite number of answers may be found. This must also be true of one equation containing three unknown quantities. Even when two such equations are given, no definite result can be obtained; for if by any of the methods

shown in Arts. 113, 114, and 115, one of the quantities be eliminated, there will still be two remaining, and only one equation.

To enable us, therefore, to solve equations of three unknown quantities, we must have three independent equations containing the same three quantities. By independent equations are meant such as cannot be derived the one from the other.

The same holds true for equations of any number of unknown quantities. If there be four such quantities, there must be four distinct and independent equations; if five, five; and so on.

The general method of solving such equations consists in eliminating one of the unknown quantities from all the equations, and doing this again with the remaining ones, until only one equation containing one unknown quantity is left.

Illustrative Examples.

(1.) Given the three equations, $x + 2y + 3z = 43$, $2x + 3y - 4z = 28$, and $5x - 4y + 2z = 38$, to find the values of x, y, and z.

Let us first eliminate z.

This may be done by "substituting," by "equating," or by "equalizing coefficients."

Adopting the last, we equalize the coefficients of z in any two of the equations, and then in one of these and the third.

The various steps of the subjoined solution will be found explained in the margin:—

$$\begin{array}{ll} & x + 2y + 3z = 43 \quad [1] \\ & 2x + 3y - 4z = 28 \quad [2] \\ & 5x - 4y + 2z = 38 \quad [3] \\ [1] \times 4, & 4x + 8y + 12z = 172 \quad [4] \end{array}$$

$$\begin{array}{lrl}
[2]\times 3, & 6x+9y-12z=\ 84 & [5]\\
[4]+[5], & 10x+17y=256 & [6]\\
[3]\times 2, & 10x-8y+\ 4z=\ 76 & [7]\\
[2]+[7], & 12x-\ 5y=104 & [8]
\end{array}$$

Having now two equations—[6] and [8]—with two u known quantities, we solve as in Art. 115:—

$$\begin{array}{lrl}
[6]\times 5, & 50x+85y=1280 & [9]\\
[8]\times 17, & 204x-85y=1768 & [10]\\
[9]+[10], & 254x=3048 & \\
 & \therefore\ x=12 &
\end{array}$$

x being known, y may be found from equations [6] [8], say [8], by substitution—

$$y = \tfrac{1}{5}(12x-104) = \tfrac{1}{5}(144-104)=8.$$

z may now be obtained from any of the original equ tions, say [1]—

$$\begin{aligned} z &= \tfrac{1}{3}(43-x-2y) = \tfrac{1}{3}(43-12-16)\\ &= \tfrac{1}{3}(43-28)=5. \end{aligned}$$

(2.) Find x, y, and z from the following equations:—

$$\frac{2}{x}+\frac{3}{y}+\frac{4}{z}=13 \qquad [1]$$

$$\frac{5}{3x}-\frac{7}{4y}+\frac{8}{z}=11 \qquad [2]$$

$$\frac{5}{x}+\frac{3}{8y}-\frac{6}{5z}=9 \qquad [3]$$

$$[2]\times 12, \quad \frac{20}{x}-\frac{21}{y}+\frac{96}{z}=132 \qquad [4]$$

$$[1]\times 24, \quad \frac{48}{x}+\frac{72}{y}+\frac{96}{z}=312 \qquad [5]$$

$$[5]-[4], \quad \frac{28}{x}+\frac{93}{y}=180 \qquad [6]$$

$[3]\times 40,\quad \frac{200}{x}+\frac{15}{y}-\frac{48}{z}=360 \quad [7]$

$[1]\times 12,\quad \frac{24}{x}+\frac{36}{y}+\frac{48}{z}=156 \quad [8]$

$[7]+[8],\quad \frac{224}{x}+\frac{51}{y}=516 \quad [9]$

$[6]\times 8,\quad \frac{224}{x}+\frac{744}{y}=1440 \quad [10]$

$[10]-[9],\quad \frac{693}{y}=924$

$$\therefore \frac{1}{y}=\frac{924}{693}=\frac{4}{3}, \text{ and } y=\frac{3}{4}.$$

Substi. in [6], $\frac{28}{x}=180-\frac{93}{y}=180-93\times\frac{4}{3}$

$$=180-124=56$$

$$\therefore \frac{1}{x}=2, \text{ and } x=\frac{1}{2}.$$

Substi. in [1], $\frac{4}{z}=13-\frac{2}{x}-\frac{3}{y}=13-2\times\frac{2}{1}-3\times\frac{4}{3}$

$$=13-4-4=5$$

$$\therefore \frac{1}{z}=\frac{5}{4}, \text{ and } z=\frac{4}{5}.$$

(3.) Find the values of x, y, and z when

$$\frac{(a+b)xy}{ab(x+y)}=\frac{1}{a-b} \quad [1]$$

$$\frac{(b+c)yz}{bc(y+z)}=\frac{1}{b-c} \quad [2]$$

$$\frac{(c+a)xz}{ac(x+z)}=\frac{1}{c-a} \quad [3]$$

$[1]\times\frac{ab}{a+b}$, $\quad \frac{xy}{x+y}=\frac{ab}{a^2-b^2} \quad [4]$

Inverting, $$\frac{x+y}{xy} = \frac{a^2-b^2}{ab} \quad [5]$$

$$\therefore \frac{1}{x}+\frac{1}{y} = \frac{a^2-b^2}{ab} \quad [6]$$

Similarly from [2], $$\frac{1}{y}+\frac{1}{z} = \frac{b^2-c^2}{bc} \quad [7]$$

And from [3], $$\frac{1}{x}+\frac{1}{z} = \frac{c^2-a^2}{ac} \quad [8]$$

[6]+[7]+[8], $$2\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\right) = \frac{a^2-b^2}{ab}+\frac{b^2-c^2}{bc}+\frac{c^2-a^2}{ac}$$

$$= \frac{a^2-b^2}{ab}+\frac{ab^2-ac^2+bc^2-a^2b}{abc}$$

$$= \frac{a^2-b^2}{ab}-\frac{ab(a-b)+(a-b)c^2}{abc}$$

$$= \frac{a-b}{abc}(ac+bc-ab-c^2)$$

$$= \frac{(a-b)(b-c)(c-a)}{abc}.$$

$$\therefore \frac{1}{x}+\frac{1}{y}+\frac{1}{z} = \frac{(a-b)(b-c)(c-a)}{2abc} \quad [9].$$

[9]−[6], $$\frac{1}{z} = \frac{(a-b)(b-c)(c-a)}{2abc}-\frac{a^2-b^2}{ab}$$

$$= \frac{a-b}{2abc}(bc-ab-c^2+ac-2ac-2b$$

$$= \frac{a-b}{2abc}(-ab-ac-bc-c^2)$$

$$= \frac{(b-a)(a+c)(b+c)}{2abc}.$$

$$\therefore z = \frac{2abc}{(b-a)(a+c)(b+c)}$$

Similarly, $$y = \frac{2abc}{(a-c)(a+b)(b+c)}$$

And $$x = \frac{2abc}{(c-b)(a+b)(a+c)}.$$

118. The method of solution shown in these examples may be extended to equations of four or any number of unknown quantities.

EXAMPLES FOR PRACTICE.—XLIII.

Find the values of the unknown quantities in the following equations:—

(1.) $2x+y+z=11$, $x-2y+z=1$, and $x+y-2z=4$.

(2.) $x-2y+3z=6$, $2x+y-2z=12$, and $3x-3y+z=2$.

(3.) $4x+3y+2z=3$, $2x+9y-z=3$, and $10x-6y+5z=3$.

(4.) $x+y+z=8$, $2x-2z=9$, and $3y-5z=5$.

(5.) $\frac{x}{3}+\frac{y}{4}=\frac{y}{2}+\frac{z}{3}=\frac{z}{2}+\frac{x}{4}=6$.

(6.) $\frac{12}{x}+\frac{12}{y}+\frac{5}{z}=5$, $\frac{9}{x}-\frac{6}{y}-\frac{4}{z}=4$, and $\frac{3}{x}+\frac{2}{y}-\frac{1}{z}=3$.

(7.) $x+y=3xy$, $x+z=4xz$, and $y+z=5yz$.

(8.) $3xy+4xz+6yz=3xyz$, $6xy-2xz-9yz=0$, and $2xy-6xz+11yz=xyz$.

(9.) $\frac{4}{x}+\frac{6}{y}+\frac{3}{2z}=4\frac{4}{5}$, $-\frac{5}{x}-\frac{9}{4y}+\frac{1}{z}=3\frac{3}{4}$, and $\frac{25}{x}-\frac{5}{y}-\frac{1}{z}=2\frac{2}{3}$.

(10.) $x-y+\frac{8}{z}=8$, $3x-4y+\frac{12}{z}=9$, and $2x+3y-\frac{240}{z}=12$.

(11.) $\frac{xy}{x+y}=1$, $\frac{xz}{x+z}=2$, and $\frac{yz}{y+z}=3$.

(12.) $(2x+y)z=6z+1$, $(7y-4x)z=z-1$, and $5(x-y)z=2(2z+1)$.

(13.) $x+y+z=a$, $x-y+z=b$, and $x+y-z=c$.

(14.) $3ax+3by-cz=1$, $-ax+2by+3cz=2$, and $2ax-by+6cz=3$.

(15.) $\frac{m}{x}+\frac{n}{y}=a$, $\frac{m}{y}+\frac{n}{z}=b$, and $\frac{m}{z}+\frac{n}{x}=c$.

(16.) $xyz=yz+xz+xy=\frac{ayz+bxz+cxy}{a+b+c}=\frac{a^2yz+b^2xz+c^2xy}{(a+b+c)^2}$.

(17.) $4u-3x=25$, $4x-5y=7$, $4y-3z=15$, $4z-u=12$.

(18.) $(3y-8)ux=4u-x$, $(4y-3z+1)x=2$,
$(5-2y+z)u=3$, $2(z-2)ux=6u+x$.

119. Problems producing Equations of more than Two Unknown Quantities.

EXAMPLES FOR PRACTICE.—XLIV.

(1.) A tradesman in three years paid income-tax to the amount of £32, 10s.; the tax being threepence in the £1 the first year, fivepence the second, and sixpence the third. If the rate had been fourpence for the first two years, he would have paid 3s. 4d. less; but if it had been fivepence for each of the three years, he would have paid £1, 13s. 4d. more than he did. What was his income each year?

(2.) There are three numbers such that one-third of the first, one-fourth of the second, and one-fifth of the third make up 30. If each be increased by 10, the sum of one-fourth of the first, one-fifth of the second, and one-sixth of the third will also be 30, as will likewise be the amount of one-half of the first, one-third of the second, and one-fourth of the third if each be diminished by 10. Find the numbers.

(3.) A, B, and C undertake a piece of work which they can together do in four days. At the close of the first day C leaves, but A and B go on for three days more, when B also leaves, and it takes A other two days to finish. Being afterwards set to a similar piece of work, C refuses to begin, while A working five days and B working six complete it between them. In what time can each man do the work separately?

(4.) A railway train and a stage-coach start at the same time from R for S. When the train has gone half way

the coach has only made 12 miles; when the train has completed its journey, the coach is three hours behind; and when the coach is 18 miles from S, the train has had time to go 21 miles beyond it. Had the coach road been 3 miles longer, by the time the coach had gone over one-third of it, the train would have been just 14 miles from its destination. Required the rate per hour of both coach and train, and the distance between the two places by road and by rail.

(5.) Three girls, D, E, and F, were seated at a table on which lay a ring, a thimble, a pencil, and a number of counters. Three of the counters were given to D, two to E, one to F, and eighteen were left on the table. One of the girls then picked up the ring and as many counters as she already had; another picked up the thimble and twice as many as she already had; while the third lifted the pencil and four times as many counters as she had given her, leaving three lying. Had this last one taken just as many as she already had, and had the one who lifted the ring taken four times what she had at first, there would have been a remainder of six counters. Name the article each girl lifted.

(6.) A certain number is composed of four digits whose sum is 24, the first and the third together being greater than the second and the fourth together by 2. If 810 be added to the number, the first three digits will each be removed one place to the left hand, and the fourth will become the first; but if 180 be subtracted, the order of the digits will be reversed. Find the number.

120. In addition to the methods of solving simultaneous equations already shown, a further one, known as that of "Indeterminate Multipliers," may be presented. It will be best explained by an example or two.

(1.) Given $\left\{\begin{array}{ll} 2x+3y=31 & [1] \\ 5x-2y=30 & [2] \end{array}\right\}$ to find x and y.

$[1]\times n$, $\quad 2nx+3ny=31n \qquad [3]$

$[2]+[3]$, $\quad (2n+5)x+(3n-2)y=31n+30 \qquad [4]$.

As the equality here expressed will not be affected any change in the value of n, we may put for it such number as will make either the coefficient of x or the efficient of y equal to nothing.

Let $3n-2=0$, the equation then becomes—

$$(2n+5)x=31n+30,$$

$$\text{And} \quad x=\frac{31n+30}{2n+5} \qquad [5].$$

From $3n-2=0$ we easily get $n=\frac{2}{3}$, and substituti this in [5], we have—

$$x=\frac{31\times\frac{2}{3}+30}{2\times\frac{2}{3}+5}=\frac{62+90}{4+15}=\frac{152}{19}=8.$$

From [2], as in former methods, we get—

$$y=\frac{1}{2}(5x-30)=\frac{1}{2}(40-30)=\frac{10}{2}=5.$$

(2.) Find the values of x and y when—

$$\frac{5}{x}+\frac{8}{y}=3 \quad [1], \text{ and } \frac{9}{x}-\frac{6}{y}=2 \quad [2]$$

$[2]\times n$, $\quad \frac{9n}{x}-\frac{6n}{y}=2n \qquad [3]$

$[1]+[3]$, $\quad \frac{9n+5}{x}-\frac{6n-8}{y}=2n+3$

Put $6n-8=0$, then $n=\frac{4}{3}$, and $\frac{9n+5}{x}=2n+3$.

$$\therefore \; x=\frac{9n+5}{2n+3}=\frac{9\times\frac{4}{3}+5}{2\times\frac{4}{3}+3}=\frac{36+15}{8+9}=\frac{51}{17}=3,$$

$$\text{And} \quad \frac{1}{y}=\frac{1}{8}\left(3-\frac{5}{x}\right)=\frac{1}{8}\left(3-\frac{5}{3}\right)=\frac{1}{8}\left(\frac{9-5}{3}\right)=\frac{1}{6}$$

$$\therefore \; y=6.$$

(3.) Find x, y, and z from the following equations :—

$$2x - 3y + 4z = 3 \quad [1]$$
$$4x - 2y - 3z = 7 \quad [2]$$
$$3x + y - 5z = 10 \quad [3]$$
$$1] \times m, \quad 2mx - 3my + 4mz = 3m \quad [4]$$
$$2] \times n, \quad 4nx - 2ny - 3nz = 7n \quad [5]$$
$$3] + [4] + [5], \quad (2m + 4n + 3)x - (3m + 2n - 1)y + (4m - 3n - 5)z = 3m + 7n + 10 \quad [6]$$

Put $3m + 2n - 1 = 0$ [7]

And $4m - 3n - 5 = 0$ [8]

Then $(2m + 4n + 3)x = 3m + 7n + 10$

$$\text{And } x = \frac{3m + 7n + 10}{2m + 4n + 3} \quad [9].$$

From [7] and [8], by any method, m may be found qual to $\frac{13}{17}$, and n to $-\frac{11}{17}$.

Substituting these in [9], we get—

$$x = \frac{3 \times \frac{13}{17} - 7 \times \frac{11}{17} + 10}{2 \times \frac{13}{17} - 4 \times \frac{11}{17} + 3} = \frac{39 - 77 + 170}{26 - 44 + 51} = \frac{132}{33} = 4.$$

Now put $2m + 4n + 3 = 0$ [10]

And $4m - 3n - 5 = 0$ [11]

Then $-(3m + 2n - 1)y = 3m + 7n + 10$

$$\text{And } y = \frac{3m + 7n + 10}{1 - 2n - 3m} \quad [12]$$

From [10] and [11], $m = \frac{1}{2}$, and $n = -1$;

$$\therefore y = \frac{\frac{3}{2} - 7 + 10}{1 + 2 - \frac{3}{2}} = \frac{3 - 14 + 20}{2 + 4 - 3} = \frac{9}{3} = 3.$$

milarly z is found to equal 1.

) Find expressions for the values of x, y, and z from quations given below :—

$$a_1x + b_1y + c_1z = d_1 \quad [1]$$
$$a_2x + b_2y + c_2z = d_2 \quad [2]$$
$$a_3x + b_3y + c_3z = d_3 \quad [3]$$

$[1]\times m$, $ma_1x+mb_1y+mc_1z=md_1$ [4]

$[2]\times n$, $na_2x+nb_2y+nc_2z=nd_2$ [5]

$[3]+[4]+[5]$, $(ma_1+na_2+a_3)x+(mb_1+nb_2+b_3)y+(mc_1+nc_2+c_3)z=md_1+nd_2+d_3$ [6]

Put $mb_1+nb_2+b_3=0$ [7]

And $mc_1+nc_2+c_3=0$ [8]

And it follows that $x=\dfrac{md_1+nd_2+d_3}{ma_1+na_2+a_3}$ [9]

$[7]\times q$, $mqb_1+nqb_2+qb_3=0$ [10]

$[8]+[10]$, $(qb_1+c_1)m+(qb_2+c_2)n+qb_3+c_3=0$ [11]

Let $qb_2+c_2=0$, then $q=-\dfrac{c_2}{b_2}$.

And $(qb_1+c_1)m+qb_3+c_3=0$ [12]

Substitute value of q in [12], then

$$\left(-\frac{b_1c_2}{b_2}+c_1\right)m+\left(-\frac{b_3c_2}{b_2}+c_3\right)=0.$$

From which $(b_1c_2-b_2c_1)m=b_2c_3-b_3c_2$,

And $m=\dfrac{b_2c_3-b_3c_2}{b_1c_2-b_2c_1}$.

In [11] let $qb_1+c_1=0$, then $q=-\dfrac{c_1}{b_1}$.

And $(qb_2+c_2)n+qb_3+c_3=0$ [13]

Substitute value of q in [13], then

$$\left(-\frac{b_2c_1}{b_1}+c_2\right)n+\left(-\frac{b_3c_1}{b_1}+c_3\right)=0.$$

This gives $(b_1c_2-b_2c_1)n=b_3c_1-b_1c_3$,

And $n=\dfrac{b_3c_1-b_1c_3}{b_1c_2-b_2c_1}$.

Now substitute these values of m and n in [9], and

$$x=\frac{md_1+nd_2+d_3}{ma_1+na_2+a_3}=\frac{\dfrac{b_2c_3-b_3c_2}{b_1c_2-b_2c_1}d_1+\dfrac{b_3c_1-b_1c_3}{b_1c_2-b_2c_1}d_2+\cdots}{\dfrac{b_2c_3-b_3c_2}{b_1c_2-b_2c_1}a_1+\dfrac{b_3c_1-b_1c_3}{b_1c_2-b_2c_1}a_2+\cdots}$$

$$=\frac{(b_2c_3-b_3c_2)d_1+(b_3c_1-b_1c_3)d_2+(b_1c_2-b_2c_1)d_3}{(b_2c_3-b_3c_2)a_1+(b_3c_1-b_1c_3)a_2+(b_1c_2-b_2c_1)a_3}.$$

milarly we may find the values of y and z; or, more ly, we may get y by changing every a into b, and b into a, in the above expression for x; and get z hanging every b into c, and every c into b, in the ession for y.

$$= \frac{(a_2c_3 - a_3c_2)d_1 + (a_3c_1 - a_1c_3)d_2 + (a_1c_2 - a_2c_1)d_3}{(a_2c_3 - a_3c_2)b_1 + (a_3c_1 - a_1c_3)b_2 + (a_1c_2 - a_2c_1)b_3}.$$

$$= \frac{(a_2b_3 - a_3b_2)d_1 + (a_3b_1 - a_1b_3)d_2 + (a_1b_2 - a_2b_1)d_3}{(a_2b_3 - a_3b_2)c_1 + (a_3b_1 - a_1b_3)c_2 + (a_1b_2 - a_2b_1)c_3}.$$

1. From these expressions we obtain for the solution uations of three unknown quantities another method d "Cross multiplication."

et us apply it to finding the values of x, y, and z in ollowing examples :—

.) $x - 2y + 3z = 11$, $4x - 3y - z = 9$, and $3x + y - 5z = 5$.

ace the equations one under the other as usual, and at the first line at the foot.

a the right hand side form three columns, headed ectively x, y, z.

		x	y	z
[1],	$x - 2y + 3z = 11$	16	-17	13
[2],	$4x - 3y - z = 9$	-7	14	-7
[3],	$3x + y - 5z = 5$	11	-13	5
[4],	$x - 2y + 3z = 11$			

Multiply the coefficient of y in [2] by the coefficient in [3]; this gives $+15$.

I. Multiply the coefficient of y in [3] by the coefficient in [2]; this gives -1.

II. Take this second product from the first, and place result, $+16$, at the end of the first line in the column x.

V. Multiply the coefficient of y in [3] by the coefficient in [4]; this gives $+3$.

V. Multiply the coefficient of y in [4] by the coefficient of z in [3]; this gives $+10$.

VI. Subtract this second product from the first, and place the remainder, -7, after the second line in column z.

VII. Deal similarly with the coefficients of y and z in [1] and [2], and place the result, which is $+11$, opposite [3], and also under x.

In every case proper attention must be paid to the effect of the signs.

These numbers, 16, -7, and 11, form multipliers by which the value of x may be found. 16 is the equivalent of $b_2c_3 - b_3c_2$ in Art. 120, No. 4; -7 of $b_3c_1 - b_1c_3$; and 11 of $b_1c_2 - b_2c_1$.

Substituting them, therefore, for these in the expression for x, we have—

$$x = \frac{11 \times 16 + 9 \times -7 + 5 \times 11}{1 \times 16 + 4 \times -7 + 3 \times 11} = \frac{176 - 63 + 55}{16 - 28 + 33} = \frac{168}{21} = 8.$$

In like manner multiply the coefficient of x in [2] by the coefficient of z in [3], and from this answer take the product of the coefficient of x in [3] and z in [2]. The result, -17, is to be placed at the end of the first line in the column y.

Obtain in the same way the numbers that are to stand at the end of the second and third lines, and substitute in the expression for y—

$$\therefore\ y = \frac{11 \times -17 + 9 \times 14 + 5 \times -13}{-2 \times -17 + -3 \times 14 + 1 \times -13} = \frac{-187 + 126 - 65}{34 - 42 - 13}$$

$$= \frac{-126}{-21} = 6.$$

By performing like operations upon the coefficients of x and y, and substituting in the expression for z, we have—

$$z = \frac{11 \times 13 + 9 \times -7 + 5 \times 5}{3 \times 13 + -1 \times -7 + -5 \times 5} = \frac{143 - 63 + 25}{39 + 7 - 25} = \frac{105}{21} = 5.$$

(2.) Solve the equations, $6x+2y-z=3$, $4x-y-2z=2$, and $2x+6y+3z=3$.

	x	y	z
$6x+2y-z=3$	9	16	26
$4x-y-2z=2$	-12	-20	-32
$2x+6y+3z=3$	-5	-8	-14
$6x+2y-z=3$			

$$x=\frac{3\times 9+2\times -12+3\times -5}{6\times 9+4\times -12+2\times -5}=\frac{27-24-15}{54-48-10}=\frac{-12}{-4}=3,$$

$$y=\frac{48-40-24}{32+20-48}=\frac{48-64}{52-48}=\frac{-16}{4}=-4,$$

$$z=\frac{78-64-42}{-26+64-42}=\frac{78-106}{64-68}=\frac{-28}{-4}=7.$$

Further practice in this method may be obtained by employing it to re-work the examples in XLIII., almost all of which may thus be solved.

CHAPTER X.

SURD EQUATIONS.

122. There is a class of equations frequently set in elementary examinations, which, on that account, it will be necessary to introduce here, although properly they should be deferred until a knowledge of surds has been acquired.

To understand the method of solving them, the following preliminary explanations must be attended to:—

123. The sign $\surd$ placed before a number or quantity indicates its square root: thus, $\surd a$ indicates the square root of a; and $\surd(a^2-x^2)$ or $\surd\overline{a^2-x^2}$ the square root of a^2-x^2. Similarly, $\sqrt[3]{}$ indicates the cube or third root, and $\sqrt[4]{}$ the fourth; so that $\sqrt[3]{a-x}$ and $\sqrt[4]{a-x}$ respectively represent the third and fourth roots of $a-x$.

124. When the root indicated cannot be exactly extracted, the expression is called a SURD. $\surd a$, $\sqrt[3]{a-x}$, and $\sqrt[4]{a^2-x^2}$ are all surds.

125. The operations of addition, subtraction, etc., may be performed on these as on other algebraic quantities,

118. The method of solution shown in these examples may be extended to equations of four or any number of unknown quantities.

EXAMPLES FOR PRACTICE.—XLIII.

Find the values of the unknown quantities in the following equations :—

(1.) $2x+y+z=11$, $x-2y+z=1$, and $x+y-2z=4$.

(2.) $x-2y+3z=6$, $2x+y-2z=12$, and $3x-3y+z=2$.

(3.) $4x+3y+2z=3$, $2x+9y-z=3$, and $10x-6y+5z=3$.

(4.) $x+y+z=8$, $2x-2z=9$, and $3y-5z=5$.

(5.) $\frac{x}{3}+\frac{y}{4}=\frac{y}{2}+\frac{z}{3}=\frac{z}{2}+\frac{x}{4}=6$.

(6.) $\frac{12}{x}+\frac{12}{y}+\frac{5}{z}=5$, $\frac{9}{x}-\frac{6}{y}-\frac{4}{z}=4$, and $\frac{3}{x}+\frac{2}{y}-\frac{1}{z}=3$.

(7.) $x+y=3xy$, $x+z=4xz$, and $y+z=5yz$.

(8.) $3xy+4xz+6yz=3xyz$, $6xy-2xz-9yz=0$, and $2xy-6xz+11yz=xyz$.

(9.) $\frac{4}{x}+\frac{6}{y}+\frac{3}{2z}=4\frac{4}{5}$, $-\frac{5}{x}-\frac{9}{4y}+\frac{1}{z}=3\frac{3}{4}$, and $\frac{25}{x}-\frac{5}{y}-\frac{1}{z}=2\frac{2}{3}$.

(10.) $x-y+\frac{8}{z}=8$, $3x-4y+\frac{12}{z}=9$, and $2x+3y-\frac{240}{z}=12$.

(11.) $\frac{xy}{x+y}=1$, $\frac{xz}{x+z}=2$, and $\frac{yz}{y+z}=3$.

(12.) $(2x+y)z=6z+1$, $(7y-4x)z=z-1$, and $5(x-y)z=2(2z+1)$.

(13.) $x+y+z=a$, $x-y+z=b$, and $x+y-z=c$.

(14.) $3ax+3by-cz=1$, $-ax+2by+3cz=2$, and $2ax-by+6cz=3$.

(15.) $\frac{m}{x}+\frac{n}{y}=a$, $\frac{m}{y}+\frac{n}{z}=b$, and $\frac{m}{z}+\frac{n}{x}=c$.

(16.) $xyz=yz+xz+xy=\frac{ayz+bxz+cxy}{a+b+c}=\frac{a^2yz+b^2xz+c^2xy}{(a+b+c)^2}$.

(17.) $4u - 3x = 25$, $4x - 5y = 7$, $4y - 3z = 15$, $4z - u = 12$.

(18.) $(3y - 8)ux = 4u - x$, $(4y - 3z + 1)x = 2$,
$(5 - 2y + z)u = 3$, $2(z - 2)ux = 6u + x$.

119. Problems producing Equations of more than Two Unknown Quantities.

EXAMPLES FOR PRACTICE.—XLIV.

(1.) A tradesman in three years paid income-tax to the amount of £32, 10s.; the tax being threepence in the £1 the first year, fivepence the second, and sixpence the third. If the rate had been fourpence for the first two years, he would have paid 3s. 4d. less; but if it had been fivepence for each of the three years, he would have paid £1, 13s. 4d. more than he did. What was his income each year?

(2.) There are three numbers such that one-third of the first, one-fourth of the second, and one-fifth of the third make up 30. If each be increased by 10, the sum of one-fourth of the first, one-fifth of the second, and one-sixth of the third will also be 30, as will likewise be the amount of one-half of the first, one-third of the second, and one-fourth of the third if each be diminished by 10. Find the numbers.

(3.) A, B, and C undertake a piece of work which they can together do in four days. At the close of the first day C leaves, but A and B go on for three days more, when B also leaves, and it takes A other two days to finish. Being afterwards set to a similar piece of work, C refuses to begin, while A working five days and B working six, complete it between them. In what time can each man do the work separately?

(4.) A railway train and a stage-coach start at the same time from R for S. When the train has gone half way,

the coach has only made 12 miles; when the train has completed its journey, the coach is three hours behind; and when the coach is 18 miles from S, the train has had time to go 21 miles beyond it. Had the coach road been 3 miles longer, by the time the coach had gone over one-third of it, the train would have been just 14 miles from its destination. Required the rate per hour of both coach and train, and the distance between the two places by road and by rail.

(5.) Three girls, D, E, and F, were seated at a table on which lay a ring, a thimble, a pencil, and a number of counters. Three of the counters were given to D, two to E, one to F, and eighteen were left on the table. One of the girls then picked up the ring and as many counters as she already had; another picked up the thimble and twice as many as she already had; while the third lifted the pencil and four times as many counters as she had given her, leaving three lying. Had this last one taken just as many as she already had, and had the one who lifted the ring taken four times what she had at first, there would have been a remainder of six counters. Name the article each girl lifted.

(6.) A certain number is composed of four digits whose sum is 24, the first and the third together being greater than the second and the fourth together by 2. If 810 be added to the number, the first three digits will each be removed one place to the left hand, and the fourth will become the first; but if 180 be subtracted, the order of the digits will be reversed. Find the number.

120. In addition to the methods of solving simultaneous equations already shown, a further one, known as that of "Indeterminate Multipliers," may be presented. It will be best explained by an example or two.

(1.) Given $\left\{\begin{array}{ll} 2x+3y=31 & [1] \\ 5x-2y=30 & [2] \end{array}\right\}$ to find x and y.

$[1]\times n$, $\quad 2nx+3ny=31n \qquad [3]$

$[2]+[3]$, $\quad (2n+5)x+(3n-2)y=31n+30 \qquad [4]$.

As the equality here expressed will not be affected by any change in the value of n, we may put for it such a number as will make either the coefficient of x or the coefficient of y equal to nothing.

Let $3n-2=0$, the equation then becomes—

$$(2n+5)x=31n+30,$$

$$\text{And} \quad x=\frac{31n+30}{2n+5} \qquad [5].$$

From $3n-2=0$ we easily get $n=\frac{2}{3}$, and substituting this in [5], we have—

$$x=\frac{31\times\frac{2}{3}+30}{2\times\frac{2}{3}+5}=\frac{62+90}{4+15}=\frac{152}{19}=8.$$

From [2], as in former methods, we get—

$$y=\frac{1}{2}(5x-30)=\frac{1}{2}(40-30)=\frac{10}{2}=5.$$

(2.) Find the values of x and y when—

$$\frac{5}{x}+\frac{8}{y}=3 \quad [1], \text{ and } \frac{9}{x}-\frac{6}{y}=2 \qquad [2]$$

$[2]\times n$, $\quad \frac{9n}{x}-\frac{6n}{y}=2n \qquad [3]$

$[1]+[3]$, $\quad \frac{9n+5}{x}-\frac{6n-8}{y}=2n+3$

Put $6n-8=0$, then $n=\frac{4}{3}$, and $\frac{9n+5}{x}=2n+3$.

$$\therefore \; x=\frac{9n+5}{2n+3}=\frac{9\times\frac{4}{3}+5}{2\times\frac{4}{3}+3}=\frac{36+15}{8+9}=\frac{51}{17}=3,$$

$$\text{And } \frac{1}{y}=\frac{1}{8}\left(3-\frac{5}{x}\right)=\frac{1}{8}\left(3-\frac{5}{3}\right)=\frac{1}{8}\left(\frac{9-5}{3}\right)=\frac{1}{6}.$$

$$\therefore \; y=6.$$

(3.) Find x, y, and z from the following equations :—

$$2x - 3y + 4z = 3 \quad [1]$$
$$4x - 2y - 3z = 7 \quad [2]$$
$$3x + y - 5z = 10 \quad [3]$$
$[1] \times m$, $$2mx - 3my + 4mz = 3m \quad [4]$$
$[2] \times n$, $$4nx - 2ny - 3nz = 7n \quad [5]$$
$[3]+[4]+[5]$, $$(2m + 4n + 3)x - (3m + 2n - 1)y + (4m - 3n - 5)z = 3m + 7n + 10 \quad [6]$$
Put $$3m + 2n - 1 = 0 \quad [7]$$
And $$4m - 3n - 5 = 0 \quad [8]$$
Then $$(2m + 4n + 3)x = 3m + 7n + 10$$
$$\text{And } x = \frac{3m + 7n + 10}{2m + 4n + 3} \quad [9].$$

From [7] and [8], by any method, m may be found equal to $\frac{13}{17}$, and n to $-\frac{11}{17}$.

Substituting these in [9], we get—

$$x = \frac{3 \times \frac{13}{17} - 7 \times \frac{11}{17} + 10}{2 \times \frac{13}{17} - 4 \times \frac{11}{17} + 3} = \frac{39 - 77 + 170}{26 - 44 + 51} = \frac{132}{33} = 4.$$

Now put $$2m + 4n + 3 = 0 \quad [10]$$
And $$4m - 3n - 5 = 0 \quad [11]$$
Then $$-(3m + 2n - 1)y = 3m + 7n + 10$$
$$\text{And } y = \frac{3m + 7n + 10}{1 - 2n - 3m} \quad [12]$$

From [10] and [11], $m = \frac{1}{2}$, and $n = -1$;

$$\therefore y = \frac{\frac{3}{2} - 7 + 10}{1 + 2 - \frac{3}{2}} = \frac{3 - 14 + 20}{2 + 4 - 3} = \frac{9}{3} = 3.$$

Similarly z is found to equal 1.

(4.) Find expressions for the values of x, y, and z from the equations given below :—

$$a_1x + b_1y + c_1z = d_1 \quad [1]$$
$$a_2x + b_2y + c_2z = d_2 \quad [2]$$
$$a_3x + b_3y + c_3z = d_3 \quad [3]$$

$[1]\times m$, $ma_1x+mb_1y+mc_1z=md_1$ [4]

$[2]\times n$, $na_2x+nb_2y+nc_2z=nd_2$ [5]

$[3]+[4]+[5]$, $(ma_1+na_2+a_3)x+(mb_1+nb_2+b_3)y+(mc_1+nc_2+c_3)z=md_1+nd_2+d_3$ [6]

Put $mb_1+nb_2+b_3=0$ [7]

And $mc_1+nc_2+c_3=0$ [8]

And it follows that $x=\dfrac{md_1+nd_2+d_3}{ma_1+na_2+a_3}$ [9]

$[7]\times q$, $mqb_1+nqb_2+qb_3=0$ [10]

$[8]+[10]$, $(qb_1+c_1)m+(qb_2+c_2)n+qb_3+c_3=0$ [11]

Let $qb_2+c_2=0$, then $q=-\dfrac{c_2}{b_2}$.

And $(qb_1+c_1)m+qb_3+c_3=0$ [12]

Substitute value of q in [12], then

$$\left(-\frac{b_1c_2}{b_2}+c_1\right)m+\left(-\frac{b_3c_2}{b_2}+c_3\right)=0.$$

From which $(b_1c_2-b_2c_1)m=b_2c_3-b_3c_2$,

And $m=\dfrac{b_2c_3-b_3c_2}{b_1c_2-b_2c_1}$.

In [11] let $qb_1+c_1=0$, then $q=-\dfrac{c_1}{b_1}$.

And $(qb_2+c_2)n+qb_3+c_3=0$ [13]

Substitute value of q in [13], then

$$\left(-\frac{b_2c_1}{b_1}+c_2\right)n+\left(-\frac{b_3c_1}{b_1}+c_3\right)=0.$$

This gives $(b_1c_2-b_2c_1)n=b_3c_1-b_1c_3$,

And $n=\dfrac{b_3c_1-b_1c_3}{b_1c_2-b_2c_1}$.

Now substitute these values of m and n in [9], and

$$x=\frac{md_1+nd_2+d_3}{ma_1+na_2+a_3}=\frac{\dfrac{b_2c_3-b_3c_2}{b_1c_2-b_2c_1}d_1+\dfrac{b_3c_1-b_1c_3}{b_1c_2-b_2c_1}d_2+d_3}{\dfrac{b_2c_3-b_3c_2}{b_1c_2-b_2c_1}a_1+\dfrac{b_3c_1-b_1c_3}{b_1c_2-b_2c_1}a_2+a_3}$$

$$=\frac{(b_2c_3-b_3c_2)d_1+(b_3c_1-b_1c_3)d_2+(b_1c_2-b_2c_1)d_3}{(b_2c_3-b_3c_2)a_1+(b_3c_1-b_1c_3)a_2+(b_1c_2-b_2c_1)a_3}.$$

Similarly we may find the values of y and z; or, more simply, we may get y by changing every a into b, and every b into a, in the above expression for x; and get z by changing every b into c, and every c into b, in the expression for y.

$$y = \frac{(a_2c_3 - a_3c_2)d_1 + (a_3c_1 - a_1c_3)d_2 + (a_1c_2 - a_2c_1)d_3}{(a_2c_3 - a_3c_2)b_1 + (a_3c_1 - a_1c_3)b_2 + (a_1c_2 - a_2c_1)b_3}.$$

$$z = \frac{(a_2b_3 - a_3b_2)d_1 + (a_3b_1 - a_1b_3)d_2 + (a_1b_2 - a_2b_1)d_3}{(a_2b_3 - a_3b_2)c_1 + (a_3b_1 - a_1b_3)c_2 + (a_1b_2 - a_2b_1)c_3}.$$

121. From these expressions we obtain for the solution of equations of three unknown quantities another method called "Cross multiplication."

Let us apply it to finding the values of x, y, and z in the following examples:—

(1.) $x - 2y + 3z = 11$, $4x - 3y - z = 9$, and $3x + y - 5z = 5$.

Place the equations one under the other as usual, and repeat the first line at the foot.

On the right hand side form three columns, headed respectively x, y, z.

		x	y	z
[1],	$x - 2y + 3z = 11$	16	−17	13
[2],	$4x - 3y - z = 9$	−7	14	−7
[3],	$3x + y - 5z = 5$	11	−13	5
[4],	$x - 2y + 3z = 11$			

I. Multiply the coefficient of y in [2] by the coefficient of z in [3]; this gives +15.

II. Multiply the coefficient of y in [3] by the coefficient of z in [2]; this gives −1.

III. Take this second product from the first, and place the result, +16, at the end of the first line in the column x.

IV. Multiply the coefficient of y in [3] by the coefficient of z in [4]; this gives +3.

V. Multiply the coefficient of y in [4] by the coefficient of z in [3]; this gives $+10$.

VI. Subtract this second product from the first, and place the remainder, -7, after the second line in column x.

VII. Deal similarly with the coefficients of y and z in [1] and [2], and place the result, which is $+11$, opposite [3], and also under x.

In every case proper attention must be paid to the effect of the signs.

These numbers, 16, -7, and 11, form multipliers by which the value of x may be found. 16 is the equivalent of $b_2c_3 - b_3c_2$ in Art. 120, No. 4; -7 of $b_3c_1 - b_1c_3$; and 11 of $b_1c_2 - b_2c_1$.

Substituting them, therefore, for these in the expression for x, we have—

$$x = \frac{11 \times 16 + 9 \times -7 + 5 \times 11}{1 \times 16 + 4 \times -7 + 3 \times 11} = \frac{176 - 63 + 55}{16 - 28 + 33} = \frac{168}{21} = 8.$$

In like manner multiply the coefficient of x in [2] by the coefficient of z in [3], and from this answer take the product of the coefficient of x in [3] and z in [2]. The result, -17, is to be placed at the end of the first line in the column y.

Obtain in the same way the numbers that are to stand at the end of the second and third lines, and substitute in the expression for y—

$$\therefore\ y = \frac{11 \times -17 + 9 \times 14 + 5 \times -13}{-2 \times -17 + -3 \times 14 + 1 \times -13} = \frac{-187 + 126 - 65}{34 - 42 - 13}$$

$$= \frac{-126}{-21} = 6.$$

By performing like operations upon the coefficients of x and y, and substituting in the expression for z, we have—

$$z = \frac{11 \times 13 + 9 \times -7 + 5 \times 5}{3 \times 13 + -1 \times -7 + -5 \times 5} = \frac{143 - 63 + 25}{39 + 7 - 25} = \frac{105}{21} = 5.$$

(2.) Solve the equations, $6x+2y-z=3$, $4x-y-2z=2$, and $2x+6y+3z=3$.

	x	y	z
$6x+2y-z=3$	9	16	26
$4x-y-2z=2$	-12	-20	-32
$2x+6y+3z=3$	-5	-8	-14
$6x+2y-z=3$			

$$x=\frac{3\times 9+2\times -12+3\times -5}{6\times 9+4\times -12+2\times -5}=\frac{27-24-15}{54-48-10}=\frac{-12}{-4}=3,$$

$$y=\frac{48-40-24}{32+20-48}=\frac{48-64}{52-48}=\frac{-16}{4}=-4,$$

$$z=\frac{78-64-42}{-26+64-42}=\frac{78-106}{64-68}=\frac{-28}{-4}=7.$$

Further practice in this method may be obtained by employing it to re-work the examples in XLIII., almost all of which may thus be solved.

CHAPTER X.

SURD EQUATIONS.

122. There is a class of equations frequently set in elementary examinations, which, on that account, it will be necessary to introduce here, although properly they should be deferred until a knowledge of surds has been acquired.

To understand the method of solving them, the following preliminary explanations must be attended to :—

123. The sign $\surd$ placed before a number or quantity indicates its square root: thus, $\surd a$ indicates the square root of a; and $\surd(a^2-x^2)$ or $\surd\overline{a^2-x^2}$ the square root of a^2-x^2. Similarly, $\sqrt[3]{}$ indicates the cube or third root, and $\sqrt[4]{}$ the fourth; so that $\sqrt[3]{\overline{a-x}}$ and $\sqrt[4]{\overline{a-x}}$ respectively represent the third and fourth roots of $a-x$.

124. When the root indicated cannot be exactly extracted, the expression is called a SURD. $\surd a$, $\sqrt[3]{\overline{a-x}}$, and $\sqrt[4]{\overline{a^2-x^2}}$ are all surds.

125. The operations of addition, subtraction, etc., may be performed on these as on other algebraic quantities,

but we shall here confine ourselves to showing the method by which multiplication is wrought.

(1.) Multiply $\sqrt{a-x}$ by $\sqrt{x}$.

The quantities $a-x$ and x are to be multiplied together, and the product placed under the root-sign; thus, $\sqrt{ax-x^2}$.

(2.) Multiply $\sqrt[3]{x+3}$ by $\sqrt[3]{(x-4)}$.

As above, the product is

$$\sqrt[3]{(x+3)(x-4)} \text{ or } \sqrt[3]{(x^2-x-12)}.$$

(3.) Multiply $\sqrt{x^2-1}$ by $\sqrt{x^2-1}$.

Here the product is $\sqrt{(x^2-1)^2}$, but the square root of $(x^2-1)^2$ is evidently x^2-1, so that

$$\sqrt{x^2-1} \times \sqrt{x^2-1} = x^2-1.$$

From this it appears that to multiply a quantity under the square root by itself, it is only necessary to remove the root-sign.

(4.) Multiply $\sqrt{x}-5$ by $\sqrt{x}+7$.

$$\begin{array}{r} \sqrt{x}-5 \\ \sqrt{x}+7 \\ \hline x-5\sqrt{x} \\ +7\sqrt{x}-35 \\ \hline x+2\sqrt{x}-35 \end{array}$$

Note.—In these examples we have not attempted to multiply together quantities under unlike root-signs, as $\sqrt{x}$ by $\sqrt[3]{y}$, and for the present they may be omitted.

126. We may now proceed to the solution of the following equations:—

(1.) $\sqrt{x}=3$.

Squaring, or multiplying each side by itself, we have $x=9$.

(2.) $\sqrt{x-2}=4$.

Squaring each side as in No. 1, $x-2=16$, $\therefore\ x=18$.

(3.) $\sqrt{x^2+11}=x+1.$

As above, $x^2+11=x^2+2x+1.$

Transposing, $2x=10,\ \therefore\ x=5.$

(4.) $\sqrt{x}+\sqrt{x+3}=\dfrac{6}{\sqrt{x+3}}.$

Multiplying by $\sqrt{x+3}$, $\sqrt{x^2+3x}+x+3=6.$

Transposing so as to get the surd, or quantity under the root, on a side by itself, $\sqrt{x^2+3x}=3-x.$

Squaring, $x^2+3x=9-6x+x^2.$

Transposing, $9x=9,\ \therefore\ x=1.$

(5.) $\dfrac{\sqrt{x}+7}{\sqrt{x}+10}=\dfrac{\sqrt{x}+1}{\sqrt{x}+2}.$

Clearing away fractions—

$$(\sqrt{x}+7)(\sqrt{x}+2)=(\sqrt{x}+1)(\sqrt{x}+10),$$
$$x+9\sqrt{x}+14=x+11\sqrt{x}+10,$$
$$2\sqrt{x}=4,\ \sqrt{x}=2,\ \therefore\ x=4.$$

This may also be solved by applying Art. 109—

$$\frac{\sqrt{x}+7}{\sqrt{x}+10}=\frac{\sqrt{x}+1}{\sqrt{x}+2}=\frac{6}{8}=\frac{3}{4}.\quad\text{(Theorem VII.)}$$
$$\frac{\sqrt{x}+7}{3}=\frac{3}{1}.\quad\text{(Theorem II.)}$$
$$\therefore\ \sqrt{x}+7=9,\ \sqrt{x}=2,\text{ and } x=4.$$

(6.) $\sqrt{8+x}+\sqrt{2+x}=2\sqrt{x-4}.$

Squaring, $8+x+2\sqrt{16+10x+x^2}+2+x=4(x-4)=4x-16.$

Collecting and transposing so as to have the surd on a side by itself—

$$2\sqrt{16+10x+x^2}=2x-26,$$
$$\sqrt{16+10x+x^2}=x-13.$$

Squaring, $16+10x+x^2=x^2-26x+169.$

Coll. and trans., $36x=153,\ \therefore\ x=4\frac{1}{4}.$

127. If $4\frac{1}{4}$ be substituted for x in the preceding equation, the first side becomes equal to 6 and the second to 1; but if the equation be written

$$\sqrt{8+x} - \sqrt{2+x} = 2\sqrt{x-4},$$

and the substitution made, both sides will be equal to one, so that the value found for x does not satisfy the equation given, but another differing from it in the sign of one of its terms.

The following four equations,

$$\begin{aligned}
\sqrt{8+x} - \sqrt{2+x} &= 2\sqrt{x-4},\\
\sqrt{8+x} + \sqrt{2+x} &= 2\sqrt{x-4},\\
-\sqrt{8+x} + \sqrt{2+x} &= 2\sqrt{x-4},\\
-\sqrt{8+x} - \sqrt{2+x} &= 2\sqrt{x-4},
\end{aligned}$$

on solution, all yield the same value of x; but on substituting this value in each of them, only the first remains an quality.

Without staying here to inquire into the reason of this, e must point out that, after solving an equation of this ass, it will be necessary to ascertain by substitution hether or not the value found is a true arithmetical ution of the equation as given.

Illustrative Example.

iven $\sqrt{x-5} = \sqrt{x}+1$, to find x.

ring, $x-5 = x+2\sqrt{x}+1$.

sposing, $2\sqrt{x} = -6$, and $\sqrt{x} = -3$.

ing again, $x = 9$.

substitution, it is found that $\sqrt{9-5} = \sqrt{9}+1$, or which is absurd. x is equal to 9 only when the n is written $\sqrt{x-5} = \sqrt{x}-1$. Solutions of this er are said to be unsatisfactory.

EXAMPLES FOR PRACTICE.—XLV.

Solve the following equations, and say when the solution is unsatisfactory :—

(1.) $\sqrt{x+13}=7$.

(2.) $\sqrt{x-8}=5$.

(3.) $\sqrt[3]{x-1}=1$.

(4.) $\sqrt{x^2+15}=x+1$.

(5.) $\sqrt{4x^2-3x-27}=2x-3$.

(6.) $\sqrt{x-7}+\sqrt{x}=\dfrac{2x-11}{\sqrt{x-7}}$.

(7.) $\sqrt{x-5}-\sqrt{x+7}=6$.

(8.) $\sqrt{5-\sqrt{x-1}}=2$.

(9.) $\dfrac{2\sqrt{x}-5}{2\sqrt{x}+1}=\dfrac{\sqrt{x}-2}{\sqrt{x}+4}$.

(10.) $\dfrac{\sqrt{3}+\sqrt{3-x}}{\sqrt{3}-\sqrt{3-x}}=\dfrac{3+\sqrt{x}}{3-\sqrt{x}}$.

(11.) $\dfrac{\sqrt{x+a}+\sqrt{x-a}}{\sqrt{x+a}-\sqrt{x-a}}=a$.

(12.) $\sqrt{x+1}+\sqrt{x-\sqrt{x+1}}=1$.

MISCELLANEOUS EXAMPLES.

Principally selected from other works.

(1.) Simplify $\frac{x^{m-1}y^{2n}}{x^{2m}y^{n+1}}$ and $\frac{\frac{x}{a}+\frac{a}{x}-2}{x-a}+\frac{\frac{x}{a}+\frac{a}{x}+2}{x+a}$.

(2.) Collect $\frac{a}{a+b}+\frac{ab}{a^2-b^2}-\frac{a^2}{a^2+b^2}$

(3.) Find the value of

$$\left(\frac{2x-a}{2x-b}\right)^2-\frac{a-x}{b-x}, \text{ when } x=\frac{ab}{a+b}.$$

(4.) Multiply together $\frac{1-a^2}{b+b^2}$, $\frac{1-b^2}{a+a^2}$, and $b+\frac{ab}{1-a}$.

(5.) Solve the equation—

$$\frac{x}{2}-\frac{x-2}{3}=\frac{1}{4}\left\{x-\frac{2}{3}(2\tfrac{1}{2}-x)\right\}-\frac{1}{3}(x-5).$$

(6.) There is a certain fish, the head of which is 9 inches long; the tail is as long as the head and half the back; and the back is as long as the head and tail together. What is the length of the fish?

(7.) Simplify $\dfrac{a^3-b^3}{a^2-b^2+\dfrac{2b^2}{1+\dfrac{a+b}{a-b}}}$.

(8.) Divide $\frac{x^2}{y^3}-\frac{1}{x}$ by $\frac{x}{y^2}+\frac{1}{y}+\frac{1}{x}$.

(9.) If $\frac{a}{b}=\frac{c}{d}$, prove that $\frac{a^2-b^2}{c^2-d^2}=\frac{ab}{cd}$.

(10.) Find the G.C.M. of

x^4+67x^2+66 and $x^4+2x^3+2x^2+2x+1$.

(11.) Add together—

$$\frac{a+b}{ax+by}, \quad \frac{a-b}{ax-by}, \text{ and } \frac{2(a^2x+b^2y)}{a^2x^2+b^2y^2}.$$

(12.) Multiply $\frac{a}{bc} - \frac{b}{ac} - \frac{c}{ab} - \frac{2}{a}$ by $1 - \frac{2c}{a+b+c}$.

(13.) Solve the equation $\frac{x-a}{x-b} = \frac{(2x-a)^2}{(2x-b)^2}$

(14.) A cistern could be filled with water by means of one pipe alone in 6 hours, and by means of another pipe alone in 8 hours; and it could be emptied by a tap in 12 hours if the two pipes were closed. In what time will the cistern be filled if the pipes and the tap are all open?

(15.) Simplify—

$$\frac{ax^m - bx^{m+1}}{a^2bx - b^3x^3} \text{ and } \frac{a^2+b^2+c^2+2ab+2ac+2bc}{a^2-b^2-c^2-2bc}.$$

(16.) Divide $\frac{x^3}{a^3} + \frac{a^3}{x^3} - 3\left(\frac{x^2}{a^2} - \frac{a^2}{x^2}\right) + \frac{x}{a} + \frac{a}{x}$ by $\frac{x}{a} + \frac{a}{x}$.

(17.) Find the L.C.M. of $x^2 - 4$, $4x^2 - 7x - 2$, and $4x^2 + 7x - 2$.

(18.) Solve $\frac{x-1}{x-2} - \frac{x-2}{x-3} = \frac{x-4}{x-5} - \frac{x-5}{x-6}$.

(19.) Split up $\frac{a^4+3a^3+2a^2+5a}{2a^4}$ into four fractions, each in its lowest terms.

(20.) Simplify $\frac{1}{a(a-b)(a-c)} + \frac{1}{b(b-a)(b-c)} - \frac{1}{abc}$.

(21.) Find the G.C.M. of $187x^3 - 84x^2 + 31x - 6$ and $253x^3 - 14x^2 + 29x - 12$.

(22.) What is the value of $\left(\frac{x-a}{x-b}\right)^3 - \frac{x-2a+b}{x+a-2b}$, when $x = \frac{1}{2}(a+b)$?

(23.) Find x from the equation $a\frac{a-x}{b} - b\frac{b+x}{a} = x$.

(24.) A cask, A, contains 12 gallons of wine and 18 gallons of water; and another cask, B, contains 9 gallons of wine and 3 gallons of water. How many gallons must be drawn from each cask so as to produce by their mixture 7 gallons of wine and 7 gallons of water?

(25.) Show that $\frac{1-x^n}{1-x} = n$, when $x = 1$.

(26.) Find a solution of $\sqrt{(x-5)} - 7 - \sqrt{(x-12)} = 0$, and say if it is satisfactory.

(27.) What is the true value of

$$\frac{ab(c^2x^2 - a^2b^2)}{c^3x^3 - a^3b^3}, \text{ when } x = \frac{ab}{c}?$$

(28.) Gather into one fraction—

$$\frac{2a^2}{b^2-4a^2} - \frac{b}{b+2a} + \frac{a}{2a-b}$$

(29.) Solve the equation $\frac{mx-a-b}{nx-c-d} = \frac{mx-a-c}{nx-b-d}$.

(30.) Multiply $\frac{x}{a} - \frac{a}{x} + \frac{y}{b} - \frac{b}{y}$ by $\frac{x}{a} - \frac{a}{x} - \frac{y}{b} + \frac{b}{y}$.

(31.) Divide $\frac{x^3}{6} - \frac{x}{4} + \frac{1}{8} - \frac{5x^2}{36}$ by $\frac{x}{3} - \frac{1}{2}$.

(32.) Reduce to its simplest form—

$$\frac{\frac{1}{1-x} - \frac{1}{1+x}}{\frac{x}{1-x} + \frac{1}{1+x}}.$$

(33.) Solve the equations—

$$a(x+y) + b(x-y) = 1. \quad a(x-y) + b(x+y) = 1.$$

(34.) Divide the number 90 into four parts, such that the first increased by 2, the second diminished by 2, the third multiplied by 2, and the fourth divided by 2, may all be equal.

(35.) If $a^2+b^2=1=c^2+d^2$, show that

$$(ad-bc)(ad+bc) = (a-c)(a+c)$$

(36.) Resolve $\dfrac{x^3-3x^2y+3xy^2-y^3}{x^2y^2}$ into four fractions, each in its lowest terms.

(37.) Simplify $\dfrac{x+y}{y} - \dfrac{x}{x+y} - \dfrac{x^2(x-y)}{y(x^2-y^2)} + \dfrac{(x-y)^2}{2xy}$.

(38.) Find x when

$$\frac{x}{10} + 10x = \frac{x}{2} + \frac{x}{5} + \frac{x}{40} - \frac{10-x}{7} + 93\tfrac{3}{4}.$$

(39.) What is the L.C.M. of $x^5+x^4y+x^3y^2+x^2y^3+xy^4+y^5$ and $x^5-x^4y+x^3y^2-x^2y^3+xy^4-y^5$?

(40.) Find the value of $\dfrac{(x^2+a^2)(x-a)}{x^n-a^n}$ when $x=a$.

(41.) A person bought a certain number of eggs, half of them at two a penny, and half of them at three a penny. He sold them again at the rate of five for twopence, and lost a penny by the bargain. What was the number of eggs?

(42.) Reduce $\dfrac{3x^3-4x^2-x-14}{6x^3-11x^2-10x+7}$ to its lowest terms.

(43.) Solve the equation—

$$\cdot 15x + \frac{\cdot 135x - \cdot 225}{\cdot 6} = \frac{\cdot 36}{\cdot 2} - \frac{\cdot 09x - \cdot 18}{\cdot 9}.$$

(44.) Show that if

$$\frac{1}{a+1} + \frac{1}{b+1} + \frac{1}{c+1} = 1, \text{ then } abc = a+b+c+2.$$

(45.) Divide $\dfrac{1}{a^3} + \dfrac{1}{b^3} + \dfrac{1}{c^3} - \dfrac{3}{abc}$ by $\dfrac{1}{a} + \dfrac{1}{b} + \dfrac{1}{c}$.

(46.) Simplify $\dfrac{x^3}{x^4-1} + \dfrac{1}{1-x-\dfrac{1}{1+x-\dfrac{1}{1-x}}}$.

(47.) Add $\dfrac{bc}{(c-a)(a-b)} + \dfrac{ca}{(a-b)(b-c)} + \dfrac{ab}{(b-c)(c-a)}$.

(48.) Find the value of x in the equation—

$$(2b+2c-x)^2+(2b-2c+x)^2-(2b-2d+x)^2=(2b+2d-x)^2.$$

(49.) It is between 11 and 12 o'clock, and it is observed that the number of minute spaces between the hands is two-thirds of what it was ten minutes previously. Find the time.

(50.) Multiply $\cdot 1x - \cdot 2y$ by $\cdot 03x + \cdot 4y$.

(51.) Find the value of $a^3 - b^3 + c^3 + 3abc$ when $a = \cdot 03$, $b = \cdot 1$, and $c = \cdot 07$.

(52.) Solve $\dfrac{c}{a-b}\left(1+\dfrac{1}{x}\right) - \dfrac{b}{a-c}\left(1+\dfrac{1}{x}\right) = \dfrac{a+c}{(a-c)x} + 1.$

(53.) If $X = b+c-a$, $Y = c+a-b$, and $Z = a+b-c$, then

$$\frac{a^2-(b-c)^2}{a^2-(b+c)^2} \times \frac{b^2-(c-a)^2}{b^2-(c+a)^2} \times \frac{c^2-(a-b)^2}{c^2-(a+b)^2} = \frac{-X \cdot Y \cdot Z}{(X+Y+Z)^3}.$$

Prove this.

(54.) Find the G.C.M. of $x^4 + 10x^3 + 20x^2 - 10x - 21$, $x^4 + 4x^3 - 22x^2 - 4x + 21$, and $x^4 - 10x^2 + 9$.

(55.) A tenant hired his farm for £10 a year in money and a corn-rent. When the corn sold at 10s. a bushel, he paid at the rate of 10s. an acre for his land; when it sold at 13s. 6d. a bushel, he paid at the rate of 13s. an acre. Find the number of bushels of corn in the rent.

(56.) Solve $\dfrac{x-a}{x-a-1} - \dfrac{x-a-1}{x-a-2} = \dfrac{x-b}{x-b-1} - \dfrac{x-b-1}{x-b-2}$.

(57.) If $s = a+b+c+\ldots$ to n terms, show that

$$\frac{s-a}{a} + \frac{s-b}{b} + \frac{s-c}{c} + \ldots = s\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\ldots\right) - n.$$

(58.) Find the sum of

$$\frac{a^2-(b-c)^2}{(a+c)^2-b^2} + \frac{b^2-(c-a)^2}{(a+b)^2-c^2} + \frac{c^2-(a-b)^2}{(b+c)^2-a^2}.$$

(59.) What is the L.C.M. of x^3-3x^2+3x-1, x^3-x^2-x+1, x^4-2x^3+2x-1, and $x^4-2x^3+2x^2-2x+1$.

(60.) Solve the equations—

$$\frac{x}{a+b}+\frac{y}{a-b}=2a. \qquad \frac{x-y}{2ab}=\frac{x+y}{a^2+b^2}.$$

(61.) A railway train, after travelling an hour, is detained 24 minutes; after which it proceeds at six-fifths of its former rate, and arrives 15 minutes late. If the detention had taken place 5 miles further on, the train would have arrived 2 minutes later than it did. Find the original rate of the train, and the distance travelled.

(62.) If $\frac{a}{b}=\frac{c}{d}=\frac{e}{f}$, prove that $\frac{a}{b}=\frac{la+mc+ne}{lb+md+nf}$.

(63.) Divide a^2+ab+b^2 by $a+a^{\frac{1}{2}}b^{\frac{1}{2}}+b$, and $x^{\frac{3}{4}}-a^3$ by $x^{\frac{1}{4}}-a^{\frac{1}{2}}x^{\frac{1}{8}}+a$.

(64.) What value of x satisfies the equation—

$$\frac{x+a}{x+b}=\left(\frac{2x+a+c}{2x+b+c}\right)^2?$$

(65.) Simplify $\left\{\frac{1+x}{1-x}+\frac{4x}{1+x^2}+\frac{8x}{1-x^4}-\frac{1-x}{1+x}\right\}$

$$\div\left\{\frac{1+x^2}{1-x^2}+\frac{4x^2}{1+x^4}-\frac{1-x^2}{1+x^2}\right\}.$$

(66.) Show that—

$$\frac{1}{x+\dfrac{1}{y+\dfrac{1}{z}}}\div\frac{1}{x+\dfrac{1}{y}}-\frac{1}{y(xyz+x+z)}=1.$$

(67.) Collect—

$$\frac{bc(a+d)}{(a-b)(a-c)}-\frac{ac(b+d)}{(a-b)(b-c)}+\frac{ab(c+d)}{(a-c)(b-c)}.$$

(68.) Solve the equations—

$$\frac{x+2y}{7}=\frac{3y+4z}{8}=\frac{5x+6z}{9}, \quad x+y-z=126.$$

(69.) A and B travel 120 miles together by rail. B, intending to come back again, takes a return ticket, for which he pays half as much again as A. They find that B travels cheaper than A by 4s. 2d. for every 100 miles. What was the price of A's ticket?

(70.) Show that $\frac{x-a}{x-b} - \frac{x-b}{x-a} = (a-b)\left(\frac{1}{a-x} + \frac{1}{b-x}\right)$.

(71.) Prove that $(ay - bx)^2 + (bz - cy)^2 + (cx - az)^2 + (ax + by + cz)^2$ is divisible by $a^2 + b^2 + c^2$ and $x^2 + y^2 + z^2$.

(72.) What values of x and y satisfy the equations—

$$2{\cdot}4x + {\cdot}32y - \frac{{\cdot}18x - {\cdot}025}{{\cdot}25} = {\cdot}8x + \frac{5{\cdot}2 + {\cdot}01y}{{\cdot}5}, \text{ and}$$

$$\frac{{\cdot}2y + {\cdot}5}{1{\cdot}5} = \frac{{\cdot}49x - {\cdot}7}{4{\cdot}2}?$$

(73.) If $s = a + b + c + \ldots$ to n terms, show that

$$\frac{s-a}{s} + \frac{s-b}{s} + \frac{s-c}{s} + \ldots = n - 1.$$

(74.) Find the time between 3 and 4 o'clock, when the hour and minute hands of a watch are—*1st.*, coincident; *2nd.*, in exactly opposite directions; and *3rd.*, at right angles to each other.

(75.) If $\frac{m}{a-b} = \frac{n}{b-c} = \frac{r}{c-a}$, show that $m + n + r = 0$.

(76.) Solve the equations—

$$3x + 5y = \frac{(8b - 2m)bm}{b^2 - m^2}, \text{ and}$$

$$b^2x - \frac{bcm^2}{b+m} + (b + c + m)my = m^2x + (b + 2m)bm.$$

(77.) Find the G.C.M. of

$$e^{2x}a^3 + e^{2x} - a^3 - 1 \text{ and } e^{2x}a^2 + 2e^xa^2 - 2e^x + a^2 - 2.$$

(78.) A and B are two towns situated 24 miles apart, on the same bank of a river. A man goes from A to B

in 7 hours, by rowing the first half of the distance and walking the second half. In returning, he walks the first half at three-fourths of his former rate; but the stream being with him, he rows at double his rate in going, and he accomplishes the whole distance in 6 hours. Find his rates of walking and rowing.

(79.) If $2s = a+b+c$, show that

$$(s-a)^2+(s-b)^2+(s-c)^2+s^2=a^2+b^2+c^2.$$

(80.) Given—

$$7yz=10(y+z),\quad 3zx=4(z+x),\quad 9xy=20(x+y),$$

to find x, y, and z.

(81.) A ship sails with a supply of biscuit for 60 days, at a daily allowance of a pound a head. After being at sea 20 days, she encounters a storm, in which 5 men are washed overboard, and damage sustained that will cause a delay of 24 days; and it is found that each man's daily allowance must be reduced to five-sevenths of a pound. Find the original number of the crew.

(82.) Reduce to its simplest form—

$$a(x+y)+\left[\left\{\frac{1}{a-x}-\frac{1}{a-y}+\frac{x}{(a-x)^2}-\frac{y}{(a-y)^2}\right\}\right.$$
$$\left.\div\left\{\frac{1}{(a-y)(a-x)^2}-\frac{1}{(a-x)(a-y)^2}\right\}\right].$$

(83.) Solve the equations—

$$\frac{\frac{7x}{4}+6y}{5}-\frac{\frac{3y+6}{5}-\frac{3x-2}{10}}{8}=\frac{80-x}{16},\text{ and}$$

$$\frac{3x}{2}+\frac{2y}{3}+\frac{5}{2}=9\left(\frac{x}{2}-\frac{y}{3}+\frac{1}{6}\right).$$

(84.) Add—

$$\frac{1}{(a-b)(a-c)(x-a)}+\frac{1}{(b-a)(b-c)(x-b)}+\frac{1}{(c-a)(c-b)(x-c)}.$$

(85.) A packet sailing from Dover with a fair wind arrives at Calais in two hours; and on its return, the wind being contrary, it proceeds six miles an hour slower than it went. Now, when it is half-way over, the wind changing, it sails two miles an hour faster, and reaches Dover in six-sevenths of the time it would have done had the wind not changed. Required the rates of sailing, and the distance between Dover and Calais.

(86.) Divide

$$a^{\frac{3}{4}} - \tfrac{3}{5}a^{\frac{7}{12}} - \tfrac{1}{8}a^{\frac{7}{10}} + \tfrac{4}{7}a^{\frac{1}{2}} + \tfrac{3}{40}a^{\frac{5}{12}} - \tfrac{1}{14}a^{\frac{5}{20}} \text{ by } a^{\frac{1}{4}} - \tfrac{1}{8}a^{\frac{1}{3}}.$$

(87.) Solve the equations—

$$x+y+z = 0, \quad (a+b)x+(a+c)y+(b+c)z = 0, \text{ and}$$
$$abx+acy+bcz = 1.$$

(88.) Prove—

$$\frac{(1+xy)(1+xz)}{(x-y)(z-x)} + \frac{(1+yx)(1+yz)}{(y-x)(z-y)} + \frac{(1+zx)(1+zy)}{(z-x)(y-z)} = 1.$$

(89.) A person sculling in a thick fog meets one barge, and overtakes another which is going at the same rate as the former. Show that if a be the greatest distance to which he can see, and b, b' the distances that he sculls between the times of his first seeing and passing the barges,

$$\frac{1}{b} + \frac{1}{b'} = \frac{2}{a}.$$

(90.) Find the G.C.M. of

$$e^{2x}x^3 + e^{2x} - x^3 - 1 \text{ and } e^{2x}x^2 + 2e^xx^2 - e^{2x} - 2e^x + x^2 - 1.$$

(91.) Solve $x - ay + a^2z = a^3$, $x - by + b^2z = b^3$, and $x - cy + c^2z = c^3$.

(92.) Prove that

$$\left(\frac{b}{c}+\frac{c}{b}\right)^2 + \left(\frac{c}{a}+\frac{a}{c}\right)^2 + \left(\frac{a}{b}+\frac{b}{a}\right)^2 = 4 + \left(\frac{b}{c}+\frac{c}{b}\right)\left(\frac{c}{a}+\frac{a}{c}\right)\left(\frac{a}{b}+\frac{b}{a}\right).$$

(93.) Given $\dfrac{ax^2+bx+c}{px^2+qx+r} = \dfrac{ax+b}{px+q}$ to find x.

(94.) If $xy+yz+xz=1$, show that

$$\frac{x}{1-x^2}+\frac{y}{1-y^2}+\frac{z}{1-z^2}=\frac{4xyz}{(1-x^2)(1-y^2)(1-z^2)}.$$

(95.) A, B, and C are employed on a piece of work. After 3 days A is discharged, one-third of the work being done. After 4 days more B is discharged, another third of the work being done. C then finishes the work in five days. In how many days could each separately do the work?

(96.) Find the value of x from the equation—

$$\frac{1}{1-\cfrac{1}{2-\cfrac{1}{3-\cfrac{1}{4-x}}}}=\frac{8}{3+\cfrac{1}{5+\cfrac{8}{x}}}.$$

(97.) Given $m\left(\frac{mn-an-bm}{am+bn-mn}\right)=n\left(\frac{m-a-b}{a+b-n}\right)$ to show that $a+b=\frac{2mn}{m+n}$.

(98.) Show that $\frac{d^m(a-b)(b-c)+b^m(a-d)(c-d)}{c^m(a-b)(a-d)+a^m(b-c)(c-d)}=\frac{b-d}{a-c}$ when $m=1$ or 2.

(99.) Solve the equation—

$$\frac{1}{(x+a)^2-b^2}+\frac{1}{(x+b)^2-a^2}=\frac{1}{x^2-(a+b)^2}+\frac{1}{x^2-(a-b)^2}.$$

(100.) Prove that $\frac{c^2+ab}{a+b}=a+b+c$ when $\frac{a^2+bc}{b+c}=\frac{b^2+ac}{a+c}$

EXAMINATION PAPERS

FOR

SCIENCE SCHOOLS AND CLASSES.

Selected from those set during last ten years.

MAY 1871.

(1.) Find the numerical value of $\frac{a(b-\sqrt{c})^2}{\sqrt{b^2-a^2}}$ when $a=3$, $b=4$, $c=5$, to two places of decimals.

(2.) Find two numbers whose sum shall be 8, and their difference ·6.

(3.) Divide $x^8-xy^7+y^8$ by x^2-xy+y^2.

(4.) Explain clearly why, in algebra, the product of a negative quantity by a negative quantity must have a positive sign.

(5.) Solve *two* of the following equations:—

(*a*) $\frac{x-5}{3}+\frac{x}{2}=12-\frac{x-10}{3}$.

(*b*) $\frac{8x+5}{14}+\frac{7x-3}{6x+2}=\frac{16x+15}{28}+\frac{2\frac{1}{4}}{7}$.

(*c*) $\sqrt{x+1}+\sqrt{x-1}=\frac{2}{\sqrt{x+1}}$.

(6.) Reduce $\frac{a+b}{ab}(a^2+b^2-c^2)+\frac{b+c}{bc}(b^2+c^2-a^2)+\frac{a+c}{ac}(a^2+c^2-b^2)$ to its simplest form.

(7.) A foreman has 116 half-crowns, and wants to pay 50 men, some at 6s. 6d. and some at 5s. 6d. each. He finds he can do this without change by paying them in groups. How many are there in each group?

MAY 1872.

(1.) If $a=4$, $b=3$, $c=11$, $x=6$, find the numerical value of $\dfrac{\sqrt{ab+4x}-\sqrt{a^2+b^2}}{\sqrt{a^2b^2-4c}-\sqrt{7x+2(ab-1)}}$.

(2.) Divide $x^5+xy^4-y^5$ by x^2-xy+y^2.

(3.) Find the least common multiple of $7(a^2-x^2)$, $9(a+x)^2$, $15(a-x)$, $3(a^2+x^2)$, $21(a^4-x^4)$.

(4.) Solve *two* of the following equations:—

(*a*) $\frac{1}{2}(5x-1)-6(22-3x)=2x-3$.

(*b*) $\dfrac{x}{x+a}-\dfrac{x-a}{x}=\dfrac{a^2}{(x-a)^2}$.

(*c*) $\left.\begin{aligned}\frac{x}{5}-\frac{y}{7}&=1\\ \frac{y}{2}-\frac{x}{3}&=2\end{aligned}\right\}$.

(5.) A train carrying three classes of passengers at 6d[illegible] 4d., and 3d., has 8 times as many third-class passengers [illegible] there are of the second-class, and 7 times as many secon[illegible] class passengers as there are of the first-class. The who[illegible]l sum received was £19, 7s. 2d. How many first-cla[illegible]s tickets were issued?

MAY 1873.

(1.) Find the values of $(bz-cy)^2+(cx-az)^2+(ay-b[illegible])^2$ and of $(a^2+b^2+c^2)(x^2+y^2+z^2)-(ax+by+cz)^2$ wh[illegible]en $a=1$, $b=2$, $c=3$, $x=1$, $y=\frac{1}{2}$, $z=\frac{1}{3}$.

(2.) Multiply $(a+b)^2+(a-b)^2$ by $(a+b)(a-b)$, a[illegible]nd divide the result by $a^3+a^2b+ab^2+b^3$.

(3.) Reduce to a single fraction the expression—

$$\frac{a+x}{a-x}+\frac{a-x}{a+x}+\frac{2ax}{a^2-2ax+x^2}+\frac{a^2+x^2}{a^2+2ax+x^2}.$$

(4.) Reduce to its simplest terms the fraction—

$$\frac{24x^3 - 82x^2 + 89x - 30}{72x^4 - 306x^3 + 469x^2 - 306x + 72}.$$

(5.) Solve *two* of the following equations—

(*a*) $\frac{34x - 56}{15} - \frac{7x - 3}{5} = \frac{7x - 5}{3} + 2\frac{1}{6}$.

(*b*) $\frac{1}{x-4} - \frac{2}{x-6} - \frac{3}{x-24} + \frac{4}{x-20} = 0$.

(*c*) $\left.\begin{matrix} ax + by = c \\ cx + ay = b \end{matrix}\right\}$.

(6.) A flock of 37 sheep, wethers and ewes, was sold for £89, 1s. 0d., a ewe fetching only three-fifths of the price of a wether. Altogether, the ewes fetched £4, 11s. 0d. more than the wethers. How many wethers were there, and what did each sell for?

(7.) Show that

$$\frac{(m-a)(n-a)}{(a-b)(a-c)} + \frac{(m-b)(n-b)}{(b-c)(b-a)} + \frac{(m-c)(n-c)}{(c-a)(c-b)} = 1.$$

MAY 1874.

(1.) Find the value, when $a = 3\frac{1}{10}$, $b = 1\frac{7}{10}$, $c = 2\frac{3}{10}$, of

$$\frac{(a+b+c)\{(b-c)^2 + (c-a)^2 + (a-b)^2\}}{a(bc - a^2) + b(ca - b^2) + c(ab - c^2)}.$$

(2.) Reduce to its simplest terms $\frac{36x^4 - 13x^2 + 1}{30x^3 - 19x^2 + 1}$.

(3.) Solve *two* of the following equations—

(*a*) $\frac{17x - 5}{18} - \frac{3x - 25}{9} = \frac{7 - 5x}{4} + 4\frac{17}{36}$.

(*b*) $\sqrt{9x + 10} - \sqrt{4x + 7} = \sqrt{(x + 11)}$.

(*c*) $ay + bx = cy + dx = 1$.

(4.) Of two square fields, one exceeds the other by 100 acres, and its side is longer than that of the other by 400 yards. Find the length of the side of each field.

(5.) A cistern has a supply cock, and also a three-way cock which can be used either for supply or discharge at pleasure. When both are used to fill it, it fills in half an hour. When the three-way cock is used to draw off, the other fills it in an hour and a half, notwithstanding the discharge going on. In what time would each fill it separately?

(6.) If $ax+by=1$, and $ax^2+by^2=\frac{1}{a+b}$, then $ax^n+by^n=(a+b)^{1-n}$.

MAY 1875.

(1.) Explain clearly the reason for the rule of signs in algebraic multiplication, by which

$$(-a)\times(-b)=+ab.$$

(2.) Find the value, when $a=2\frac{1}{2}$, $b=3\frac{1}{2}$, $c=4\frac{1}{2}$, of

$$\frac{b^2c^2(c^2-b^2)+c^2a^2(a^2-c^2)+a^2b^2(b^2-a^2)}{(b+c)(c+a)(a+b)}.$$

(3.) Find the least common multiple of

$1-8x+17x^2+2x^3-24x^4$ and $1-2x-13x^2+38x^3-24x^4$.

(4.) Solve *two* of the following equations—

(*a*) $\dfrac{4x-3}{6x-5}=\dfrac{3\frac{1}{7}}{6-3\frac{1}{8}}$.

(*b*) $(x+a)(x+3a)(x+6a)=(x+2a)(x+4a)^2$.

(*c*) $\left\{\begin{matrix} x+\sqrt{xy}=18 \\ x-y=9 \end{matrix}\right\}$.

(5.) A lady, wishing to relieve a number of poor people, finds that, if she gives them 1s. a piece, she will have 3s. 4d. left, and that she has not enough by 2s. 4d. to give them 1s. 4d. a piece. How many had she to relieve, and how much money had she to distribute?

(6.) If $\dfrac{1}{a}+\dfrac{1}{b}+\dfrac{1}{a+x}=0$, and $\dfrac{1}{a}+\dfrac{1}{c}+\dfrac{1}{a+y}=0$, also $\dfrac{1}{a}+\dfrac{1}{x}+\dfrac{1}{y}=0$, then $a+b+c=0$.

MAY 1876.

(1.) Find the value, when $x=3$, $y=4$, $z=5$, of $x^2+(z-y)(z+y)+\frac{x-z}{x+z}(x-y-z)(x+y-2z)$.

(2.) Find the least common multiple of $24ax^2y^3$, $45a^2x^2y^2$, $75a^2x^3y$, and $100a^3xy^2$.

(3.) Show that $a-b$, $b-c$, and $c-a$ cannot be all three positive or all three negative.

(4.) Solve *two* of the following equations—

(*a*) $\frac{3x-7}{5}-\frac{3x+7}{4}=\frac{5x-9}{8}-\frac{3x+9}{6}$.

(*b*) $\frac{x-a}{x+a}+\frac{3b-x}{2b+x}=0$.

(*c*) $\left\{\begin{array}{rl} x+3y= & 19 \\ 3x-y= & 7 \end{array}\right\}$.

(5.) Two arm-chairs cost as much as five common chairs, and I bought a set of three arm-chairs and fourteen ordinary chairs for £37, 12s. 6d. What was the cost of each?

(6.) One square of carpet contains 16 more square yards than another, and its side is 2 yards longer than the side of the other. How much does each measure?

MAY 1877.

(1.) Find the value of

$$(b-c)^3+2(c-a)^3+(a-b)^3-3(b-c)(c-a)(a-b),$$

when $a=1$, $b=-\frac{1}{2}$, $c=\frac{3}{2}$.

(2.) Divide $5a^4-18a^3x+5a^2x^2-x^4$ by $x^2+3ax-a^2$.

(3.) Simplify the expression—

$$ac-b^2)(ce-d^2)+(ae-c^2)(bd-c^2)-(ad-bc)(be-ca).$$

(4.) Reduce to its lowest terms $\frac{x^4+4x^3-19x^2-46x+120}{x^4-25x^2+144}$.

(5.) Solve *two* of the following equations—

$$(a)\ x - \frac{3}{5} - \frac{5(x-2)}{4} = \frac{3}{2}\left(x - \frac{1}{10}\right).$$

$$(b)\ a(x-a) + b(x-b) + 2ab = 0.$$

$$(c)\ \left\{\begin{array}{l} 2x = 9 - 3y \\ 5y = 24 - 6x \end{array}\right\}.$$

(6.) At what time between 4 and 5 o'clock are the hands of a watch exactly at right angles to one another?

MAY 1878.

(1.) Find, when $a=1$, $b=3$, $c=4$, the value of

$$\frac{(ac-bc)(a+b) + bc(c-a) - ca(a-b)}{(b-c)(c-a)(a+b)}.$$

(2.) Simplify the expression—

$$\frac{2(x^2+2ax+a^2)(x^2-2ax+a^2) - 2(a^2-x^2)^2 + 5ax(a+x)^2 - 20a^2x^2}{(x-a)(x+a)}.$$

(3.) Find the greatest common measure of

$$45x^3y + 3x^2y^2 - 9xy^3 + 6y^4 \text{ and } 54x^2y - 24y^3.$$

(4.) Solve *two* of the following equations—

$$(a)\ \frac{7x-28}{3} - 3\tfrac{1}{4} + \frac{4x-21}{7} = x - 7\tfrac{1}{4} - \frac{9-7x}{8}.$$

$$(b)\ \frac{b}{a} - \frac{dx}{c} = \frac{ax}{b} - \frac{c}{d}.$$

$$(c)\ 2x + 3y = 3x + 2y = 25.$$

(5.) The depth of a pond at one end is twice as great as at the other. Eighteen inches of water (in depth) are drained off, and the deep end is then three times as deep as the shallow end. What were the original depths?

(6.) Find the value of $x^3 - 8y^3 + 29z^3 + 18xyz$, when $2y = x + 3z$ and $z = 5$.

MAY 1879.

If m and n are positive whole numbers, explain—
he meaning of A^m; *secondly*, why $A^m \times A^n = A^{m+n}$;
mplify $\dfrac{27a^{m+p}b^{n-q}}{81a^m b^n}$.

When $a=3$, $b=4$, $c=-5$, what is the value of

$$a^2+b^2-c^2) + \frac{b+c}{bc}(b^2+c^2-a^2) + \frac{c+a}{ca}(c^2+a^2-b^2)?$$

Reduce to its lowest terms—

$$\frac{x^4-2x^3-25x^2+26x+120}{x^4-4x^3-19x^2+46x+120}.$$

Solve *two* of the following equations—

(*a*) $\frac{1}{3}(x-7\frac{1}{2})+\frac{4}{9}(x-15)=\frac{1}{2}(4x-11)$.

(*b*) $\dfrac{1}{x-3}-\dfrac{1}{x-7}=\dfrac{1}{x-2}-\dfrac{1}{x-6}$.

(*c*) $ax+by=bx-ay=c$.

A herd of 125 cattle was sold for £2575. There
ıalf as many oxen again as there were cows, and the
:etched altogether £25 less than the cows. What
.e price of each ox and of each cow?

If one part of £400 is put out at 4 per cent. per
ı, and the other part at 5 per cent. per annum, and
yearly income obtained is £18, 5s., what are the

MAY 1880.

Find the value, when $x=5$ and $y=3$, of

$$\frac{x^4-4x^3y+6x^2y^2-5xy^3+2y^4}{2x^4-5x^3y+6x^2y^2-4xy^3+y^4}$$

Multiply a^3-x^3 by a^2-x^2, and divide the product
$-x)^2$.

3.) Simplify $(a-b)(b+c)(c+a)+(b-c)(c+a)(a+b)+(c-a)(a+b)(b+c)$, and find its value when $a=1$, $b=3$, and $c=-2$.

(4.) Solve the equations—

$$(a)\ 7(x+\tfrac{1}{3})-5x\left(\frac{1}{3x}+\frac{1}{2\frac{1}{2}}\right)=4.$$

$$(b)\ \left\{\begin{array}{l}0{\cdot}5x+0{\cdot}07y=0{\cdot}93\\0{\cdot}03x-0{\cdot}4y=0{\cdot}46\end{array}\right\}.$$

(5.) The rent of a shop is two-sevenths of the rent of the whole house of which it is a part. Being separately rated, its occupier pays £10, 15s. a year less in rates than the occupier of the rest of the house. The rates are 3s. 7d. in the £. What is the rent of the whole house?

(6.) Find, as a fraction in its lowest terms, the value of

$$\frac{1}{x^3-3x^2-15x+25}-\frac{1}{x^3+7x^2+5x-25}.$$

ANSWERS.

EX. I.—(Page 14.)

$4a+17b-15c$.
$8a^2-17ab+25b^2$.
$6x^2+19xy+20y^2$.
$-15x^3-13x^2y^2-14y^3$.
$a^2-12b^2+8c^2$.
a^2b-3bc^2.

(7.) $24x+4y$.
(8.) $19x^3-9x^2-31x+3$.
(9.) $8a+6b-7x-5z$.
(10.) $7a^2+3a-7x+10x^2$.
(11.) $12a^4-a^3b+5a^2b^2-8ab^3+4b^4$.
(12.) x^4-4x^3-4x+1.

The terms need not be in the same order as here set down, but they must otherwise be the same.

EX. II.—(Page 15.)

$4a^2+3ab-2b^2)x$.
$m-5n)y$.
$x^3-3a^2+3a-1)x^2$.
$2a-b-c)yz$.
$4-ab)z^2$.
$ax+ay$.
$1+a+a^2)x^3+(a^2+a+1)x^2$.

(8.) $(4m-2p+2)xy$.
(9.) $(2a^2+2b^2)x+(2a-2b)y$.
(10.) $(a^3-a^2b+ab^2-b^3)x$
$+(a^3+2a^2b+2ab^2+b^3)y$
$+(a^3-a^2b^2+b^3)z$.
(11.) $(b^3-c^3)x+(a^2-c^3)x^2+(a+c)x^3$.
(12.) $(5a+3b)x+(4a-b)z$.

EX. III.—(Page 17.)

4.
36.

(3.) 0.
(4.) -7.

(5.) 1.
(6.) 0.

EX. IV.—(Page 19.)

(3.) $1\frac{1}{2}$.
(4.) 3.

(5.) 13.
(6.) 37.

EX. V.—(Page 21.)

(1.) 12; 24; 36.
(2.) 6 feet; 24 feet.
(3.) 6 years; 30 years.
(4.) 12; 84.
(5.) 23.
(6.) 8 quarts.
(7.) 8s. 4d.; 6s. 8d.; 5s.
(8.) 10 hours; 190 miles.
(9.) £180; £120; £60.
(10.) 50 miles.
(11.) 8 horses; 8 oxen; 16 sheep.
(12.) £1, 1s.

EX. VI.—(Page 26.)

(1.) $3axy$.
(2.) $11a^2$.
(3.) $3ab$.
(4.) $-5xy^2$.
(5.) $-7abc$.
(6.) $-8am$.
(7.) $5a-3b$.
(8.) $9a^3+7abc$.
(9.) $3a^2-b^2$.
(10.) $-2xy+8y^2$.
(11.) $3a^2-2b^2$.
(12.) $4a+2b-4c$.
(13.) $2x^2-3y^2-2a^2+3b^2$.
(14.) $-3ax+3by+3cz$.
(15.) $4x^4+2x^3-8x^2+2x-6$.
(16.) c^2+d^2.
(17.) $a^2-4a^2b+a+4ab^2$.
(18.) $b+d-y-z$ or $y+z-b-d$.
(19.) $(4a^2-5bc)x^2$.
(20.) $(3a+2b-3c)xy$.
(21.) $(m^2+2mn-3n^2)y^2$.
(22.) $(ab-de)x+(bc+ef)y+(cd-fg)z$
(23.) $(4a^2+b^2-3)x+(2b^2-4a^2-3)y$
(24.) $(2a^2-4ab+2b^2)x^2$.

EX. VII.—(Page 29.)

(1.) $3a-3x$.
(2.) $2x^2-x$.
(3.) $4a^2+8b^2$.
(4.) $3x^3-4x^2+4x-3$.
(5.) $4a^2+2$.
(6.) $2a+2b$.
(7.) 1.
(8.) $-2y^3$.
(9.) $2b^3-2ab^2$.
(10.) $5x^4+2x^3-2x^2+2x-2$.
(11.) $4a+5b$.
(12.) 0.

EX. VIII.—(Page 30.)

(1.) 4. (2.) 11. (3.) 47. (4.) 6. (5.) 0. (6.) –

EX. IX.—(Page 31.)

(1.) $(3a-5b+4c)$.
(2.) $(-4a^2+2ab-b^2)$.
(3.) $-(ax+by-cz)$.
(4.) $-(-a^3+a^2-2a+2)$.
(5.) $(a+b)-(c+d)+(e-f)$.
(6.) $(a+b-c)-(d-e+f)$.
(7.) $(3a+4b)+(3c-4x)-(3y+4$
(8.) $(3a+4b+3c)-(4x+3y+4z$

(9.) $\{5a^3-(3a^2+2a)\}-\{4b^3-(2b^2-b)\}$.
(10.) $\{x^3+(2x^2-3x)\}-\{y^3-(2y^2+3y)\}$.
(11.) $-[-\{5a^3-(3a^2+2a)\}+\{4b^3-(2b^2-b)\}]$, or
$-[\{4b^3-(2b^2-b)\}-\{5a^3-(3a^2+2a)\}]$.
(12.) $-[-\{x^3+(2x^2-3x)\}+\{y^3-(2y^2+3y)\}]$, or
$-[\{y^3-(2y^2+3y)\}-\{x^3+(2x^2-3x)\}]$.

EX. X.—(Page 34.)

.	(4.) 5.	(7.) 3.
.	(5.) 3.	(8.) $\frac{1}{2}$.
.	(6.) 6.	(9.) $\frac{3}{8}$.
(10.) 3.		
(11.) 4.		
(12.) 7.		

EX. XI.—(Page 36.)

34 ; 293.
1 feet ; 18 feet.
30 women.
feet 1 inch ; 3 feet 4 inches.
67 ; £32, 10s. ; £15, 5s.
1.
1 boys ; 355 nuts.
) years ; 20 years.
15 sheep ; £138.
(10.) $27\frac{1}{2}$ miles.
(11.) 2 P.M.
(12.) 4s. 6d.
(13.) 15 ; 12 ; 8.
(14.) 36 ; 28 ; 4.
(15.) 99 ; 75.
(16.) $3\frac{1}{2}$ hours.
(17.) 37 hits.
(18.) 18 years ; 24 years ; 30 years.

EX. XII.—(Page 44.)

$4a^2b^2cx^2yz$.
$5a^4b^4m^4x^2$.
$\cdot 36c^4x^3y^4z^3$.
$\cdot 55m^5n^3x^2y^3$.
$\frac{1}{2}p^4q^3r^5s^3$.
$73a^3b^3c^3x^4y^4z^4$.
$2a^3b^2+20a^2b^3$.
$4a^3xy^2-6a^2by^3$.
$x^3bxy-12a^2b^2xy+12ab^3xy$.
$b^2+13xy+6y^2$.
$^3+3a^2b+3ab^2+b^3$.
$x^4-3a^3x-6a^2x+2ax^2-15ax^3+5x^4$.
(13.) $16a^4+4a^2x^2+x^4$.
(14.) $9x^4y^4-4x^2y^2+4xy-1$.
(15.) $a^4-8a^3b+24a^2b^2-32ab^3+16b^4$.
(16.) $-7x^2+50xy-7y^2$.
(17.) $a^2-b^2-2bc-c^2$.
(18.) $16x^8+7x^6+6x^4+3x^2+1$.
(19.) $x^3+y^3+z^3-3xyz$.
(20.) $4a^4-20a^3+33a^2-20a+4$.
(21.) x^3-4x^2+x+6.
(22.) $a^6m^6-a^4m^4-a^2m^2+1$.
(23.) a^8-16x^8.
(24.) $ab+cd$.

$^3+(2a+b)x^2+(a^2+2ab)x+a^2b$.
$^3-(3a+2b)x^3+(3a+6ab+2b)x^2-(9a^2+4b^2)x+6ab$.
$^3x^3+(a^2b-a^2c+a^2d)x^2-(abc-abd+acd)x-bcd$; or, $a^3x^3+(b-c+d)a^2x^2-(bc-bd+cd)ax-bcd$.
$^3-(a-c)x+(a-b)(b-c)$, or $x^2-(a-c)x+ab-ac-b^2+bc$.
$^4-2mx^3+(2b^2-1+m^2)x^2-2b^2mx+b^4$.
$^3+2bx^5+(a^2+3b^2)x^4+(4a^2b+4b^3)x^3-(a^4+2a^2b^2-3b^4)x^2+(2a^4b-4a^2b^3+2b^5)x-(a^2-b^2)^3$.
$^3+2bx^5+(a^2+3b^2)x^4+4(a^2+b^2)bx^3-(a^2-b^2)(a^2+3b^2)x^2+2(a^2-b^2)^2bx-(a^2-b^2)^3$.

EX. XIII.—(Page 50.)

$9a^2+28abc+4b^2c^2$.
$5x^4-10x^2y^2+y^4$.
$^2+4y^2+9z^2+4xy+6xz+12yz$.
(4.) $x^4-4x^3+6x^2-4x+1$.
(5.) x^4-49.
(6.) x^4+4x^2+16.

(7.) $x^4-x^2y^2+2xy^3-y^4$.
(8.) x^3-1.
(9.) $m^4x^4-2m^2n^2x^2y^2+n^4y^4$.
(10.) $(a+b)^2-(c+d)^2$, or $a^2+2ab+b^2-c^2-2cd-d^2$.
(11.) $(1-x^3)^2-(2x-2x^2)^2$, or $1-4x^2+6x^3-4x^4+x^6$.
(12.) $1+8x^3$.
(13.) $a^3b^3-c^3$.
(14.) $a^4-8a^3+16a^2-16$.
(15.) $-36x^2+108x$.
(16.) 0.
(17.) $a^4x^4+a^2b^2x^2y^2+b^4y^4$.
(18.) x^8+81x^4+6561.
(19.) $35x^2y^2$.
(20.) $84x+28$.
(21.) $30x^2+210$.
(22.) $2x^3-x^2-6x$.
(23.) $(3a-3b)x$.
(24.) $2x^3+6a^2b+2b^3$.

EX. XIV.—(Page 51.)

(1.) 259.
(2.) 123.
(3.) 303.
(4.) 1.
(5.) 14.
(6.) 3272.

EX. XV.—(Page 52.)

(1.) 4.
(2.) 2.
(3.) 5.
(4.) 3.
(5.) 7.
(6.) $5\frac{1}{4}$.
(7.) $1\frac{2}{3}$.
(8.) 8.
(9.) 1.
(10.) 2.
(11.) $1\frac{1}{4}$.
(12.) 1.

EX. XVI.—(Page 56.)

(1.) 8.
(2.) 35; 5.
(3.) 7; 8.
(4.) 23 yards; 46 yards.
(5.) 60 feet; 50 feet; 1800 sq. feet.
(6.) 15 gallons; 7 gallons; 10 gallons.
(7.) 17; 11.
(8.) 18 men; 21 women; 6 children.
(9.) 5d.
(10.) 169.
(11.) 665.
(12.) $3\frac{1}{2}$ miles.
(13.) 2820 gallons.
(14.) 5 gallons.
(15.) 18; 23; 162 miles.
(16.) 30 feet.
(17.) 117.
(18.) 19 oxen; 380 sheep; £912.

EX. XVII.—(Page 69.)

(1.) $2a^2c^2$.
(2.) $-\frac{4a^2x}{y}$.
(3.) $-\frac{3}{2}a^3c$.
(4.) $\frac{8}{9}\frac{1}{ax^2y^2}$.
(5.) $2a^3-3a^2x+4ax^2$.
(6.) $\frac{3}{2}a^2-\frac{9}{4}ax+3x^2$.
(7.) $\frac{2x}{3a}+1-\frac{4a}{3x}$.
(8.) $\frac{7}{d}+\frac{8}{a}-\frac{9}{b}-\frac{10}{c}$.
(9.) $-\frac{amx^2}{3}+\frac{4xy}{3}-\frac{2y^2}{am}$.

-1.
$+3y$.
$^2-3xy+y^2$.
$-3a^2x^2-2x^4$.
$^2+2x-1$.
$-2x^2+x-2$.
$+2+\frac{3}{2x-3}$.
$^3-x^2+3x-1+\frac{2x+1}{2x^2+x+1}$.
$-3-\frac{2x^2-5x-2}{x^3-2x^2-2x+1}$.
$-2y-3z$.
$^2-2xy+y^2+2x+y+1$.
$+2xy+y^2+xz+yz+x+y+$
z^2+2z+1.

(22.) $1-x^2+x^4-x^6+$ etc.
(23.) $1+2x-2x^3-2x^4+2x^6+2x^7-$
$2x^9-$ etc.
(24.) $1-3x^2+2x^3+3x^4-6x^5+2x^6+$
etc.
(25.) $x^2-ax+ab$.
(26.) x^2-bx+a.
(27.) $x^2+(a-b)x-ab$.
(28.) $1+(a-b)x+(a+b)x^2-abx^3$.
(29.) $1+(1+a)x+(a+b)x^2$
$+(a+b+c)x^3+(b+c)x^4$
$+(c+1)x^5+x^6$.
(30.) $a-(a-b)x-(b-c)x^2$
$+(a-c+d)x^3-(a-b+d)x^4$
$-(b-c)x^5+\frac{(a-c+d)x^6+(b-c)x^7}{1+x+x^2}$

EX. XVIII.—(Page 75.)

$c(a^2+bc)$.
$c+y)(ax-y)$.
$(a+b)(a-b)x$.
$(a+2x)(a-2x)x$.
$x+1)^2$.
$1-3a)^2$.
$c+1)x^2$.
$+y+z)(x+y-z)$.
$+y-z)(x-y+z)$.
$-2b+3c)(a-2b-3c)$.
$+n+p+q)(m-n+p-q)$.
$+b)^2(a-b)^2$.
$+3)(x+4)$.
$-2)(x-5)$.
$+5)(x-2)$.
$-4)(x+1)$.
$+12)(x+1)$.

(18.) $(x+4)(x-3)$.
(19.) $(x+12)(x+5)$.
(20.) $(x-20)(x+3)$.
(21.) $(x-a)(x-c)$.
(22.) $(x-a)(x+b)$.
(23.) $(ax-by)(ax-cy)$.
(24.) $(ax+by)(cx-dy)$.
(25.) $(4x-3y)(3x-4y)$.
(26.) $(6x-5)(4x+3)$.
(27.) $(x+1)(x+y)$.
(28.) $(x+y)(x+y+1)$.
(29.) $(x+1)(x^2-2x-1)$.
(30.) $(x-3)(x+2)^2$.
(31.) $(2x+3y-4z)(2x-3y+4z)$.
(32.) $(x-1)^2(x^2+x+1)(x^2+x-1)$.
(33.) $(2a+3x)(4a^2-6ax+9x^2)$.
(34.) $(x^2+y^2)(x^4-x^2y^2+y^4)$.

(35.) $(x+y)(x-y)(x^2-xy+y^2)(x^2+xy+y^2)$.
(36.) $2a(2a-5x)(4a^2+10ax+25x^2)$.
(37.) $(x+1)(x+y)(x^2-xy+y^2)$.
(38.) $(4x-1)(16x^2+4x+1)$.
(39.) $(ax^2+b^3y)(a^2x^4-ab^3x^2y+b^6y^2)$.
(40.) $(x^2+1)(x+1)(x-1)(y^2+1)(y+1)(y-1)$.
(41.) $(4+2x+x^2)(4-2x+x^2)$.
(42.) $(a^2b^2+3ab+9)(a^2b^2-3ab+9)$.

(43.) $(x^4-x^2y^2+y^4)(x^2-xy+y^2)(x^2+xy+y^2)$.
(44.) $(x^2+4)(x+2)(x-2)$.
(45.) $(x^2+2x+2)(x^2-2x+2)$.
(46.) $(x^2+2x+2)(x^2-2x+2)(x^2+2)(x^2-2)$.
(47.) $(x+y)(x+y+z)(x+y-z)$.
(48.) $(4x+1)(x-2y)(x^2+2xy+4y^2)$.
(49.) $(x+y)(x-y)^2(x^2+xy+y^2)$.
(50.) $2(x-y)(x-1)(y-1)$.

EX. XIX.—(Page 76.)

(1.) $2x^2$.
(2.) 1.
(3.) $2(x^2-x-1)$.
(4.) $a+c$.
(5.) $4x(2y+1)$.
(6.) $2x^2$.

EX. XX.—(Page 80.)

(1.) x^3-4x^2+5x-6.
(2.) $4a^6b^3-8a^2b-16$.
(3.) $3+4x^2+5x^4+6x^6$.
(4.) $1+4x+3x^2+2x^3+x^4$.
(5.) $x^3-3x^2-4-\dfrac{2x+5}{4x^2+3x-2}$.
(6.) $10x^4-5x^3y+5xy^3-y^4-\dfrac{15x^2y^5-8xy^6}{5x^3+3xy^2-y^3}$.

EX. XXI.—(Page 82.)

(1.) x^{m+n}; $x^{m+1}y^{n+1}$.
(2.) a^{x+y+z}.
(3.) $a^{3m}-a^{2m}x^n+a^mx^{2n}-x^{3n}$.
(4.) $x^{m+2}-x^my^2+x^2y^m-y^{m+2}$.
(5.) $a^2x^{5m}-acx^{3m+1}+(ac-b^2)x^{3m}-(ad-bc)x^{2m+1}+bcx^{2m}-(bd+c^2)x^{m+1}+cdx^2$.
(6.) $x^m-(m-1)^m$.
(7.) $x^{m+1}-x^m+x^{m-1}$.
(8.) $x^{m+1}y^m+x^2y^{2m-1}$.
(9.) a^m-x^n.
(10.) $a^n-a^{n-1}+a^{n-2}$.
(11.) $ax^m-bx^{m-1}+cx^{m-2}$.
(12.) $x^n-nx^{n-1}+n(n-1)x^{n-2}$.

EX. XXII.—(Page 83.)

(1.) $a^{-1}+x^{-1}$.
(2.) $x-a+x^{-1}a^2-x^{-2}a^3$.
(3.) $x^3-ax^2+a^2x-a^3+a^4x^{-1}$.
(4.) $x^{-4}-ax^{-3}+bx^{-2}-cx^{-1}+d$.
(5.) $a^{-m}+a^{-(m+1)}+a^{-(m+2)}$.
(6.) $x^{2n}+x^n+1+x^{-n}+x^{-2n}+$ etc.

EX. XXIII.—(Page 84.)

(1.) $\dfrac{3}{10}$.
(2.) $\dfrac{4}{9}$.
(3.) $2\frac{3}{4}$.
(4.) $\dfrac{641}{10000}$
(5.) 3.
(6.) $\dfrac{21^3}{100}=92{\cdot}61$.

EX. XXIV.—(Page 86.)

(1.) $x^3+4x^2+16x+64$.
(2.) $x^4-2x^3+4x^2-8x+16$.
(3.) $27x^3-9x^2+3x-1+\dfrac{2}{3x+1}$.
(4.) $32x^5-48x^4+72x^3-108x^2+162x-243$.
(5.) $25a^6-35a^3b^2+49b^4-\dfrac{686b^6}{5a^3+7b^2}$.

EX. XXV.—(Page 87.)

(1.) 3.	(7.) 4.	(13.) 1.
(2.) $\frac{1}{2}$.	(8.) 1.	(14.) $1\frac{1}{3}$.
(3.) 1.	(9.) 10.	(15.) 1.
(4.) 2.	(10.) 3.	(16.) 4.
(5.) 1.	(11.) 4.	(17.) 2.
(6.) 3.	(12.) $2\frac{1}{2}$.	(18.) 3.

MISCELLANEOUS EXAMPLES.—(Page 89.)

1.) $a^2+b^2+c^2$.
2.) $3x^n+2x^{n-1}+2x^{n-2}-3x^{n-3}$.
3.) 384.
4.) 12.
5.) a.
6.) 25.
7.) $a^2b^2+c^2d^2-a^2c^2-b^2d^2$.
8.) $1-(1+a)x+(1+a+b)x^2-(1+a+b+c)x^3+$ etc.
9.) $a+3b+4x+4y$.
10.) $2x$.
11.) 2d. per score.
12.) $a^2c^2+b^2d^2+a^2d^2+b^2c^2=a^2(c^2+d^2)+b^2(c^2+d^2)$, $(-2\mp2)^2+(-1\pm4)^2=25=(1+4)(4+1)$.
13.) $x^3+2x+1+7x^{-1}+3x^{-2}$.
14.) $x(x-6)(3x+4)$.
15.) 3.
16.) £2, 10s.
17.) $(m^2-n^2)x^4+2(m-n)qx^3+2(m-n)x^2$.
18.) $2ax-2by$.
19.) $m^4x^4-13m^2n^2x^2y^2+36n^4y^4$.
20.) $2x$.
21.) $a^2+4ab-12b^2$.
22.) 72 hours.
23.) $bx+ay+(a+b)z$.
24.) $a-\{b-c-(d-e-\overline{f-g})\}$.
25.) $4x^2$.
26.) $mn(m-n)$.
27.) 2.
28.) 100 bushels.
29.) 0.
30.) When $a-b=1$.
31.) $x^2+(a-p)x+a^2-ap+q+\frac{a^3-a^2p+aq-r}{x-a}$.
32.) a^2.
33.) $(x^2+y)(x^4-x^2y+y^2)(x^2+y^2)$.
34.) $2x+1=\overline{x+1}+x$.
35.) 3 inches; 9 inches.
36.) $a+b+c+d$.
37.) $ac=b^2$, $\therefore (a+c)^2-b^2=a^2+b^2+c^2$.
38.) By Art. 81, $\frac{7^{2n+1}+1^{2n+1}}{7+1}$ has no remainder.
39.) a^2.
40.) $x^{2(a+b+c)}$; $a^0b^0c=c$.
41.) 2 inches; $1\frac{1}{3}$ inch.
42.) 0.
43.) $4(a^2+b^2+c^2)$.
44.) $x^6+x^5y-x^3y^3-x^2y^4+y^6$.
45.) $(8a^2+2b^2)\times4ab=8ab(4a^2+b^2)$.
46.) $11\frac{1}{4}$d.
47.) $(2a+c)x+(2b+a)y+(2c+b)z$.
48.) x^3.
49.) $5\frac{1}{2}$.
50.) $\frac{(a+b)(ab+bc+ac)+abc+bc^2+ac^2-abc}{a+b}=\frac{(a+b)(b+c)(a+c)}{a+b}$
$=(b+c)(a+c)$.
51.) 640.

(52.) $ab+ac-bc$.

(53.) $\{(a+b)^2-c^2\}\{(a-b)^2+c^2\}=(a^2-b^2)^2+$ etc.

(54.) $(x-ab)(a-b)+bx-b^2c-cx+bc^2+cx-c^2a-ax+ca^2$
$=(x-ab)(a-b)+(a^2-b^2)c-(a-b)x-(a-b)c^2$
$=(a-b)(x-ab+ac+bc-x-c^2)=(a-b)(b-c)(c-a)$.

(55.) 5.

(56.) 7s. and 11s. 8d.

(57.) $n^3-p^3-3(n^2-p^2)+2(n-p)$ etc.

(58.) $a^{-(m+n)}+1$.

(59.) Put $a+b=-c$, $b+c=-a$, $c+a=-b$, $\therefore (a+b)(b+c)(c+a)$
$=-abc$, $(a+b+c)^3=a^3+b^3+c^3+3a^2(b+c)+3b^2(a+c)+$
$3c^2(a+b)+6abc=0$.

(60.) Resolve into the factors $(a-b)(a-c)(b-c)$.

(61.) £2000.

(62.) Arrange $(1+a)^2-(1+c)^2+(c+ac)^2-(a+ac)^2$.

(63.) $x^2y-x^2z+y^2x+y^2z-z^2x-z^2y=(x+y)(x+z)(y-z)$.

(64.) $\frac{4}{13}$.

(65.) $(n-x)^2-x^2=n^2-2nx=n\{(n-x)-x\}$.

(66.) 55 miles from B.

(67.) $\frac{a}{x^2}-\frac{1}{x}+\frac{1}{a^2-x^2}-\frac{x}{a^2(a^2-x^2)^3}+a^2x^2$.

(68.) $(a-b)(b+c)(c+a)+(a+b)(bc-c^2+ab-ac+bc+c^2-ab-a$
$=(a-b)(bc+ab+c^2+ac-2ac-2bc)=$ etc., -20.

(69.) $x^2-y^2-y^{-2}+x^{-2}=x^2+2+x^{-2}-(y^2+2+y^{-2})$ etc.;
$=x^2-y^2-x^2x^{-2}y^{-2}+x^{-2}y^2y^{-2}=$ etc.

(70.) $x^4-2x^2+1-x^2=x^4-x^3+3x^2-x+2+x^3-6x^2+11x-6$. $\therefore x=$

(71.) $(a+b)^3+3(a+b)^2c+3(a+b)c^2+c^3-(a^3+b^3+c^3)$
$=(a+b)\{(a+b)^2+3(a+b)c+3c^2-(a^2-ab+b^2)\}$ etc.

(72.) Let x = number B left, A cut off 48, B cut off 28.

EX. XXVI.—(Page 98.)

(1.) a^2b^2.
(2.) $4axy^2$.
(3.) $5x^2y^2z^2$.
(4.) $19m^3n^3$.
(5.) $4x^2$.
(6.) abc^2.
(7.) $2x-y$.
(8.) $x(x-2y)$.
(9.) $x^2(x-y)$.
(10.) $x+1$.
(11.) $m-x$.
(12.) $x^2(x+2)$.
(13.) $x+y+z$.
(14.) $3x-2y+z$.
(15.) $(a+1)(x+1)-$
(16.) $a-x$.
(17.) $x^{n-2}y^{n-1}$.
(18.) $ab+ac+bc$.

EX XXVII.—(Page 104.)

(1.) $x-1$.
(2.) x^3-x^2-x+1.
(3.) $x-3$.
(4.) x^2+x-2.
(5.) x^2+2x+3.
(6.) $x+2$.
(7.) $x(x^2+5)$.
(8.) $x(x-4)$.
(9.) $3x(3x^2+2x+1)$.
(10.) x^2-x+1.
(11.) $x(x^m+yx^{m-1}-1)$.
(12.) $ax^{-2}-bx^{-1}+c$.

EX. XXVIII.—(Page 106.)

(1.) $x-2$. (2.) $5x^2+3x-2$. (3.) x^3-1.

EX. XXIX.—(Page 109.)

(1.) $\frac{mnxy}{mn}$; $\frac{mnx^2+mny^2}{mn}$.

(2.) $\frac{a^4-2a^2x^2+x^4}{(a+x)^2}$.

(3.) $\frac{x^4-1}{x-1}$.

(4.) $\frac{x^3+y^3+z^3-3xyz}{x+y+z}$.

(5.) $\frac{x^{m+n}-x^my^{2m}}{x^n+y^m}$.

(6.) $\frac{1-2x+2x^2-x^3}{b-a}$.

EX. XXX.—(Page 112.)

(1.) $\frac{3xy}{4ac}$.

(2.) $\frac{19yz}{20mx}$.

(3.) $\frac{7a}{3x-5y}$.

(4.) $\frac{1-2x+3x^2}{x^3}$.

(5.) $\frac{x-2y}{2x+y}$.

(6.) $\frac{5x+4y}{7y-6z}$.

(7.) $\frac{x-4}{x-5}$.

(8.) $\frac{y-z}{y+z}$.

(9.) $\frac{a}{b}$.

(10.) $\frac{a+x}{a+2x}$.

(11.) $\frac{a-x}{a}$.

(12.) $\frac{2}{m^2y+2}$.

(13.) $\frac{a^2+xy}{a^4+a^2xy+x^2y^2}$.

(14.) $\frac{2x-y-2z}{2x+y-2z}$.

(15.) $\frac{x+y}{x+y^2}$.

(16.) $\frac{p+q}{r+s}$.

(17.) $\frac{x-y}{x+y}$.

(18.) $\frac{2x-3}{2x+3}$.

(19.) $\frac{x+2}{x^2+4}$.

(20.) $\frac{x-1}{x+1}$.

(21.) $\frac{2x^2-3x+2}{3x^2-2x+3}$.

(22.) $\frac{a-x}{b-x}$.

(23.) $\frac{x-3}{x(x+2)}$.

(24.) $\frac{mx+n}{px-q}$.

EX. XXXI.—(Page 119.)

(1.) $\frac{5a^2c^2x^2y^2}{2bz}$.

(2.) $\frac{24(x-2)}{x(x+2)}$.

(3.) $\frac{b^2(a-x)}{a^2(b-x)}$.

(4.) $\frac{2x(x-1)}{x+1}$.

(5.) $\frac{x-1}{x+a}$.

(6.) $\frac{2x}{3y}$.

(7.) $\frac{x^2-2x+3}{x^2-2x-3}$.

(8.) $\frac{1}{x^2}$.

(9.) 1.

(10.) $\frac{b+c}{b-x}$.

(11.) $\frac{a-2c}{a-2b}$.

(12.) $-\frac{a-b+c+1}{a-b-c+1}\cdot\frac{x-y+z-1}{x-y-z-1}$.

(13.) $\frac{2}{7}x^2y^2z$.

(14.) $\frac{x^4z^2}{5a^4b^2c^2}$.

(15.) $\frac{acmnx^2}{bpqy^2}$.

(16.) $\frac{a-b}{a+b}$.

(17.) $\frac{x-2}{x}$.

(18.) $x+1$.

(19.) $\frac{x-3}{x-4}$.

(20.) $\frac{x^4+x^2+1}{x^2-1}$.

(21.) $\frac{2}{1+x+x^2+x^3}$.

(22.) 1.

(23.) $\frac{5}{3}$.

(24.) $x^{2n+2}y(x^2-y^2)^{2n-2}(x^2+y^2)^2$.

EX. XXXII.—(Page 123.)

(1.) $120a^3m^3xyz$.

(2.) $ab^2c^2defx^2y^2z^2$.

(3.) $180m^2x^3y^2z$.

(4.) $abc(b+c)$.

(5.) $420(x^2-y^2)$.

(6.) $x^2(x+1)(x^3-1)$, or $x^6+x^5-x^3-x^2$.

(7.) $(x^2-4)(x^2-9)$.

(8.) $4x^4-5x^2+1$.

(9.) $x^2(x+3)(x-7)(x+8)$.

(10.) x^6-1.

(11.) $(x^4-1)(x^2+x+1)^2$.

(12.) $(a+b+c)(a-b+c)(a+b-c)(x^2-y^2)$.

EX. XXXIII.—(Page 125.)

(1.) $x^4+3x^3-3x^2-7x+6$.

(2.) $6x^5+13x^4-4x^3-18x^2-2x+5$.

(3.) $6x^5+x^4y-43x^3y^2-43x^2y^3+xy^4+6y^5$.

EX. XXXIV.—(Page 129.)

(1.) $\frac{3a}{30}$; $\frac{2b}{30}$.

(2.) $\frac{75x}{120}$; $\frac{70y}{120}$; $\frac{54z}{120}$.

(3.) $\frac{4bcx^2}{2abcx^3}$; $\frac{3acx}{2abcx^3}$; $\frac{8ab}{2abcx^3}$.

(4.) $\frac{(a^2+b^2)x}{abx}$; $\frac{b(x^2-a^2)}{abx}$; $\frac{a(x^2-b^2)}{abx}$.

(5.) $\frac{ab(a-b)^2}{ab(a^2-b^2)}$; $\frac{a^2(a^2-b^2)}{ab(a^2-b^2)}$; $\frac{b^2(a^2-b^2)}{ab(a^2-b^2)}$; $\frac{ab(a+b)^2}{ab(a^2-b^2)}$.

(6.) $\frac{3(x-3)}{(x-1)(x-2)(x-3)}$; $\frac{2(x-2)}{(x-1)(x-2)(x-3)}$; $\frac{x-1}{(x-1)(x-2)(x-3)}$.

(7.) $\frac{x^3-3x+2}{(x-1)(x-2)(x-3)(x+4)}$; $\frac{2x^2-8x+6}{(\ )(\ )(\ )(\ )}$; $\frac{2x^2-10x+12}{(\ )(\ )(\ )(\ )}$; $\frac{x^2+3x-4}{(\ )(\ )(\ )(\ )}$.

(8.) $\frac{aa(x-a)(x-2b)}{(x-a)(x-b)(x+2a)(x-2b)}$; $\frac{2ba(x+2a)(x-b)}{(\ )(\ )(\ )(\ )}$; $\frac{3ab(x-a)(x-b)}{(\ )(\ )(\ )(\ )}$.

(9.) $\frac{x^3+x}{x(x^3+8)}$; $\frac{x^3-2x+4}{x(x^3+8)}$; $\frac{x^3-4x}{x(x^3+8)}$.

(10.) $\frac{x^6-ax^5+a^2x^3-a^5x+a^6}{x^8+a^4x^4+a^8}$; $\frac{x^6+ax^5-a^3x^3+a^5x+a^6}{x^8+a^4a^4+a^8}$; $\frac{x^4+a^2x^2+a^4}{x^8+a^4a^4+a^8}$.

(11.) $\frac{(x+1)(3x-2)}{(2x+3)(3x+2)(3x-2)}$; $\frac{x(2x+3)}{(\ )(\ )(\ )}$; $\frac{(x-1)(3x+2)}{(\ )(\ )(\ )}$; $\frac{(4x^2-9)(9x^2-4)}{(\ )(\ )(\ )}$.

(12.) $\frac{a^n(b+1)^n(a-1)}{b^2(b+1)^n(a-1)}$; $\frac{b^{n+1}(a-1)}{b^2(\ )(\ )}$; $\frac{(b+1)^{n-1}(a^2-1)}{b^2(\ )(\ )}$; $\frac{b^2(b+1)^{n+1}}{b^2(\ )(\ )}$; $\frac{b^2(b+1)^{n+2}(a-1)^2}{b^2(\ )(\ )}$.

EX. XXXV.—(Page 133.)

(1.) $\frac{3x+1}{3x-1}$; $\frac{10y-5}{10y+2}$; $\frac{15-8x}{16+6x}$.

(2.) $\frac{4}{2a-1}$; $-\frac{2a+3x}{6}$; $\frac{x}{y}$.

(3.) $\frac{1}{a^2-x^2}$; $\frac{x-1}{x+1}$.

(4.) $\frac{2xy}{x^2+y^2}$; $\frac{x^2(x-y)}{y^2(x+y)}$.

(5.) $\frac{13}{30}$; $\frac{24ab^2+6b}{24a^2b^2+18ab+1}$.

(6.) $\frac{x^2+1}{x^3+1}$; $\frac{x^4-x^3-1}{x^5+x^2}$.

EX. XXXVI.—(Page 137.)

(1.) $\frac{169a}{72}$.

(2.) $\frac{2ab^2c}{35}$.

(3.) $\frac{2a+3b+2c}{abc}$.

(4.) $\frac{a^2-x^2}{abx}$

(5.) $\frac{x^2}{a(x-a)}$.

(6.) $\frac{a}{x+a}$.

(7.) 2.

(8.) $\frac{x+y}{x-y}$.

(9.) 1.

(10.) $\frac{b^2}{a(b-a)}$.

(11.) $\frac{1}{(x-1)(x-2)(x+2)}$.

(12.) $\frac{3}{4(2x+1)}$.

(13.) $\frac{2x^2}{x^2-1}$.

(14.) $\frac{x^2}{y^2-x^2}$.

(15.) $\frac{3x}{(x+1)(x+2)(x-3)}$.

(16.) 0.

(17.) $\frac{9x}{x^3-27}$.

(18.) $\frac{2x^3}{x^4+4x^2+16}$.

(19.) $\frac{4a^4(2x^2+a^2)}{x^6-a^6}$.

(20.) $\frac{4x^7}{x^8-16}$.

(21.) $\frac{14x^2}{(16x^2-1)(9x^2-1)}$.

(22.) 1.

(23.) 0.

(24.) $\frac{a-b}{a+b}$.

(25.) $\frac{2y}{y-x}$.

(26.) $1-2x$.

(27.) 0.

(28.) $\frac{(a+b)^2}{2ab}$.

(29.) $\left\{\frac{(x-y)(y-z)(x-z)}{2xyz}\right\}^2$.

(30.) $\frac{a^4+x^4}{a^3+x^3}$.

(31.) $\frac{z}{x}$.

(32.) $\frac{2x^3y^3}{x^4y^4+x^2y^2+1}$.

(33.) 1.

(34.) $\frac{a^2c^2+b^4}{a^2b^2+b^2c^2}$.

(35.) $a+b+c$.

(36.) $2(a^n+b^n+c^n)$.

EX. XXXVII.—(Page 141.)

(1.) $\frac{101}{210}$.

(2.) $\frac{3}{2}$.

(3.) 4

(4.) 0.

(5.) $\frac{(a-b)(a-2b)}{a^2}$.

(6.) $a+b$.

(7.) 0.

(8.) $\frac{10}{43}, \frac{2}{7}, \frac{0}{1}, -2$.

(9.) $\frac{1-b}{(a-2)(b-2)}$.

(10.) $\frac{c^2}{b^2}$.

(11.) $\frac{(s-b)(s-c)}{s(s-a)}$

(12.) $a+b+c$.

EX. XXXVIII.—(Page 145.)

(1.) $\frac{1}{3}$.

(2.) 0.

(3.) 1.

(4.) $\frac{3}{4}$.

(5.) 2.

(6.) ∞.

EX. XXXIX.—(Page 151.)

(1.) 2.

(2.) 6.

(3.) 15.

(4.) 5.

(5.) 4.

(6.) 3.

(7.) -3.

(8.) 7.

(9.) 8.

(10.) 10.

(11.) 4.

(12.) $2\frac{1}{2}$.

(13.) $\frac{3}{4}$.

(14.) 4.

(15.) -10.

(16.) 5.

(17.) 6.
(18.) $\frac{3}{19}$.
(19.) 1.
(20.) 3.
(21.) 4.
(22.) -1.
(23.) 8.
(24.) 3.
(25.) $\frac{11}{18}$.
(26.) 6.
(27.) $-5\frac{1}{2}$.
(28.) 2.
(29.) $\frac{1}{ab}$.
(30.) $\frac{ab}{a+b}$.
(31.) 1.
(32.) $\frac{m^2-1}{m}$.
(33.) 2.
(34.) $\left(\frac{a+1}{a-1}\right)^2$.
(35.) $\frac{a^2+b^2}{a^2+b^2}$.
(36.) $\frac{(a-b)c}{ab-ac-bc}$.

EX. XL.—(Page 157.)

..) 20.
:.) 156 gallons.
:.) 48; 45.
..) 15 hours.
:.) 21s.; 14s.; 5s. 3d.
:.) 42s.
:.) 6 miles; $5\frac{1}{2}$ miles.
:.) 8s.
:.) £3800; £2800.
:.) 32; 28; 24; 21.
..) A, 14d.; B, 14d.; C, 7d.
:.) 1440 miles.
:.) $32\frac{8}{11}$ minutes past 6.
(14.) $21\frac{6}{11}$ minutes and $54\frac{6}{11}$ minutes past 7.
(15.) $5\frac{5}{11}$ minutes past 7.
(16.) 3 times.
(17.) 48 minutes past 10.
(18.) 12 minutes past 2.
(19.) £540.
(20.) 190 and 185.
(21.) 22 guineas; 26 guineas.
(22.) 29 miles.
(23.) 1080 yards; $16\frac{1}{2}$ minutes.
(24.) £700; £100.

EX. XLI.—(Page 169.)

(1.) 5; 2.
(2.) 4; 1.
(3.) 7; 2.
(4.) 12; 5.
(5.) 8; 9.
(6.) 10; 6.
(7.) 8; -2.
(8.) 6; 8.
(9.) 15; 8.
(10.) 11; 3.
(11.) 6; 50.
(12.) 20; 8.
(13.) 3; 2.
(14.) 2; 10.
(15.) 5; -3.
(16.) 7; 5.
(17.) 9; -8.
(18.) 2; 1.
(19.) 5; 4.
(20.) $\frac{1}{2}$; $\frac{1}{2}$.
(21.) $\frac{c(a^2-bc)}{a(ab-c^2)}$; $\frac{b(a^2-bc)}{a(ac-b^2)}$.
(22.) $\frac{a+b}{a-b}$; 1.
(23.) $\frac{b^2}{ac}$; $\frac{a^2}{bc}$.
(24.) $\frac{1}{a^2-1}$; $\frac{1}{1-b^2}$.

EX. XLII.—(Page 173.)

..) £50; £20.
:.) $\frac{11}{12}$.
:.) 10 yards; 20 yards.
:.) $4\frac{1}{2}$ miles; $3\frac{1}{3}$; 3.
(5.) 17 guineas; 13 sovereigns; 11 crowns.
(6.) $\frac{a}{n+1}$; $\frac{a}{m+1}$; $\frac{a(mn-1)}{(m+1)(n+1)}$.

(7.) 25 persons; £7, 10s.
(8.) $3\frac{1}{3}$ miles; $2\frac{1}{2}$ miles.
(9.) 345.
(10.) For A, 640 and 460. For B, 480 and 540.
(11.) 19.
(12.) David spent 20s.; David's wife, 24s.; George, 6s.; George's wife, 10s. As 10s. is one-sixth of the whole, this was spent by Mary; she therefore was George's wife, and Jane was David's.
(13.) 47; 11.
(14.) 72 apples; 60 pears.
(15.) 14 feet long, 10 feet broad.
(16.) 150 yards; A, 30 yards; B, 20 yards.
(18.) Dog, 450; hare, 600.
(19.) 50 miles; 30 miles.
(20.) 2s. 6d.
(21.) 15 miles; 2 miles.
(22.) 10; 18s.
(23.) Expenditure, £60. Produce —1st year, £90; 2nd year, £120.
(24.) 21 of each.

EX. XLIII.—(Page 183.)

(1.) 4; 2; 1.
(2.) 7; 8; 5.
(3.) $\frac{1}{4}$; $\frac{1}{3}$; $\frac{1}{2}$.
(4.) 5; $2\frac{1}{2}$; $\frac{1}{2}$.
(5.) 12; 8; 6.
(6.) 3; 2; -1.
(7.) 1; $\frac{1}{2}$; $\frac{1}{3}$.
(8.) 6; 4; 3.
(9.) 5; -3; $\frac{1}{4}$.
(10.) 18; 12; 4.
(11.) $1\frac{1}{7}$; $2\frac{2}{7}$; -12.
(12.) $2\frac{1}{2}$; $1\frac{1}{2}$; 2.
(13.) $\frac{1}{2}(b+c)$; $\frac{1}{2}(a-b)$; $\frac{1}{2}(a-c)$.
(14.) $\frac{1}{6a}$; $\frac{1}{3b}$; $\frac{1}{2c}$.
(15.) $\frac{m^3+n^3}{am^2-bmn+cn^2}$; $\frac{m^3+n^3}{bm^2-cmn+an^2}$; $\frac{m^3+n^3}{cm^2-amn+bn^2}$.
(16.) $\frac{(a-b)(a-c)}{(a+b)(a+c)}$; $\frac{(b-a)(b-c)}{(b+a)(b+c)}$; $\frac{(c-a)(c-b)}{(c+a)(c+b)}$.
(17.) 16; 13; 9; 7.
(18.) 1; 2; 3; 4.

EX. XLIV.—(Page 184.)

(1.) £500; £540; £600.
(2.) 30; 40; 50.
(3.) 10; 12; 15 days.
(4.) Coach 9 miles, train 21 miles, per hour. Distance by coach-road 51 miles, by railroad 56 miles.
(5.) F lifted the ring; D the thimble; E the pencil.
(6.) 6756.

EX. XLV.—(Page 196.)

(1.) 36.
(2.) 33.
(3.) 2.
(4.) 7.
(5.) 4.
(6.) 16.
(7.) 9; unsatisfactory.
(8.) 2.
(9.) 9.
(10.) $2\frac{1}{4}$.
(11.) $\frac{1}{2}(a^2+1)$.
(12.) 3; unsatisfactory.

MISCELLANEOUS EXAMPLES.—(Page 197.)

$\frac{x-1}{x+1}$; $\frac{2}{a}$.

$\frac{x^2b^2}{x^2-b^4}$.

$\frac{x-b}{x}$.

inches.

-1.

$\frac{a^2}{c^2}=\frac{b^2}{d^2}$.

$+1$.

$\frac{x^4x^3-b^4y^3)}{x^4x^4-b^4y^4}$.

$\frac{(a-c)^2-b^2}{abc}$.

$\frac{x}{x+b}$.

hours.

$\frac{x^{m-1}}{x+bx)}$; $\frac{a+b+c}{a-b-c}$.

$-3\left(\frac{x}{a}-\frac{a}{x}\right)+\frac{a^2}{x^2}$.

$6x^2-1)(x^2-4)$.

$+\frac{3}{2a}+\frac{1}{a^2}+\frac{5}{2a^3}$.

$\frac{1}{c(a-c)(b-c)}$.

$x-3$.

$-b$.

from A; 4 from B.

$+x+x^2+\ldots\ldots+x^{n-1}$ to n terms.

(26.) 21; unsatisfactory.

(27.) $\frac{2}{3}$.

(28.) $\frac{ab-b^2}{b^2-4a^2}$.

(29.) $\frac{a+b+c+d}{m+n}$.

(30.) $\frac{x^2}{a^2}+\frac{a^2}{x^2}-\frac{y^2}{b^2}-\frac{b^2}{y^2}$.

(31.) $\frac{x^2}{2}+\frac{x}{3}-\frac{1}{4}$.

(32.) $\frac{2x}{1+x^2}$.

(33.) $\frac{1}{a+b}$; 0.

(34.) 18; 22; 10; 40.

(35.) Put $a^2=1-b^2$, and $d^2=1-c^2$, and multiply.

(36.) $\frac{x}{y^2}-\frac{3}{y}+\frac{3}{x}-\frac{y}{x^2}$.

(37.) $\frac{1}{2}\left(\frac{x}{y}+\frac{y}{x}\right)$.

(38.) 10.

(39.) x^6-y^6.

(40.) $\frac{2}{na^{n-3}}$.

(41.) 60.

(42.) $\frac{x^2+x+2}{2x^2+x-1}$.

(43.) 5.

(45.) $\frac{1}{a^2}-\frac{1}{ab}-\frac{1}{ac}+\frac{1}{b^2}-\frac{1}{bc}+\frac{1}{c^2}$.

(46.) $\frac{x^2}{1-x^4}$.

(47.) -1.

(48.) $c+d$.

(49.) 40 minutes past 11.

(50.) $\cdot003x^2+\cdot034xy-\cdot08y^2$.

(51.) 0.

(52.) $\frac{a^2-b^2-c(b-c)}{b(b-c)-(a-c)^2}$.

(54.) x^2-1.

(55.) 120 bushels.

(56.) $\frac{1}{2}(a+b+3)$.

(57.) $1+1+1 \ldots$ to n terms $= n$.

(58.) 1.

(59.) $(x-1)^3(x+1)(x^2+1)$.

(60.) $(a+b)^2$; $(a-b)^2$.

(61.) 25 miles; $47\frac{1}{2}$ miles.

(63.) $a-a^{\frac{1}{2}}b^{\frac{1}{2}}+b$, $x^{\frac{1}{2}}+a^{\frac{1}{2}}x^{\frac{3}{8}}-a^{\frac{3}{8}}x^{\frac{1}{8}}-a^2$.

(64.) $\frac{c^2-ab}{a+b-2c}$.

(65.) $\frac{2(1+x^4)}{x}$.

(67.) d.

(68.) 51; 76; 1.

(69.) 20s.

(71.) Factors are—
$(a^2+b^2+c^2)(x^2+y^2+z^2)$.

(72.) 10; 5.

(74.) $16\frac{4}{11}$, $49\frac{1}{11}$, and $32\frac{8}{11}$ minutes past 3.

(76.) $\frac{bm}{b-m}$; $\frac{bm}{b+m}$.

(77.) e^x+1.

(78.) 4 miles walking; 3 miles rowing at first; $4\frac{1}{2}$ miles in still water.

(80.) 4; 5; 2.

(81.) 40.

(82.) $2a^2$.

(83.) 4; 3.

(84.) $\frac{1}{(x-a)(x-b)(x-c)}$.

(85.) Rates, 11, 5, 7 miles; distance, 22 miles.

(86.) $a^{\frac{1}{2}}-\frac{3}{5}a^{\frac{1}{3}}+\frac{4}{7}a^{\frac{1}{4}}$.

(87.) $\frac{1}{(b-c)(a-c)}$; $\frac{1}{(a-b)(c-b)}$; $\frac{1}{(a-b)(a-c)}$.

(90.) $(e^x+1)(x+1)$.

(91.) abc; $ab+ac+bc$; $a+b+c$.

(93.) $\frac{br-cq}{cp-ar}$.

(95.) A, 36; B, 60; C, 15 days.

(96.) 1.

(99.) $\frac{a^2+b^2}{a+b}$.

EXAMINATION PAPERS.

MAY 1871.—(Page 207.)

(1.) 3·528.

(2.) 4·3; 3·7.

(3.) $x^6+x^5y-x^3y^3-x^2y^4+y^6$.

(4.) Explain as in Art. 43.

(5.) (*a*) $14\frac{4}{7}$; (*b*) 1; (*c*) 1.

(6.) $2(a+b+c)$.

(7.) 15 at 6s. 6d.; 35 at 5s. 6d.

MAY 1872.—(Page 208.)

(1.) $\frac{1}{2}$.

(2.) $x^3+x^2y-y^3$.

(3.) $315(a+x)(a^4-x^4)$.

(4.) (*a*) 7; (*b*) $\frac{a}{3}$; (*c*) 15, 14.

(5.) 23 first-class.

MAY 1873.—(Page 208.)

(1.) $10\frac{1}{15}$; $10\frac{1}{15}$.

(2.) $2(a-b)$.

(3.) $\dfrac{3a^4+6a^2x^2-x^4}{(a^2-x^2)^2}$.

(4.) $\dfrac{4x-5}{12x^2-25x+12}$.

(5.) (a) $-2\frac{21}{41}$; (b) $\frac{13}{23}$; (c) $\dfrac{ac-b^2}{a^2-bc}$, $\dfrac{ab-c^2}{a^2-bc}$.

(6.) 13 wethers at 65s. each.

(7.) Reduce to L.C.D.

MAY 1874.—(Page 209.)

(1.) -2.

(2.) $\dfrac{6x^2+5x+1}{5x+1}$.

(3.) (a) 2; (b) $-\dfrac{61}{67}$, unsatisfactory; (c) $\dfrac{a-c}{ad-bc}$, $\dfrac{d-b}{ad-bc}$.

(4.) 805 yards; 405 yards.

(5.) $\frac{3}{4}$ hour; $1\frac{1}{2}$ hour.

(6.) $x=y=\dfrac{1}{a+b}$.

MAY 1875.—(Page 210.)

(2.) 2.

(3.) $(1+x)(1-4x)(1-2x)(1-3x)(1+4x)(1-x)$, or $1-5x-11x^2+85x^3-86x^4-80x^5+96x^6$.

(4.) (a) $\dfrac{397}{412}$; (b) $\dfrac{-14a}{5}$; (c) 12, 3.

(5.) 17 persons; 20s. 4d.

(6.) Find the values of x and y from first and second equations, and substitute in third.

MAY 1876.—(Page 211.)

(1.) $13\frac{1}{2}$.

(2.) $1800a^3x^3y^3$.

(4.) (a) $-\dfrac{21}{11}$; (b) $\dfrac{ab}{2a-5b}$; (c) 4, 5.

(5.) 35s.; 87s. 6d.

(6.) 5 yards; 3 yards.

MAY 1877.—(Page 211.)

(1.) $\dfrac{1}{8}$.

(2.) $-x^2+3ax-5a^2$.

(3.) $a^2cd-acd^2-abc^2-bc^2d+b^2d^2+c^4$.

(4.) $\dfrac{x^2+3x-10}{x^3-x-12}$.

(5.) (a) $\dfrac{41}{35}$; (b) $\dfrac{(a-b)^2}{a+b}$; (c) $3\frac{3}{8}$, $\frac{3}{4}$.

(6.) $5\frac{5}{11}$ minutes; and $38\frac{2}{11}$ minutes past 4.

MAY 1878.—(Page 212.)

(1.) -1.

(2.) $\dfrac{5ax(x-a)}{x+a}$.

(3.) $3(3x+2y)y$.

(4.) (*a*) 7; (*b*) $\dfrac{bc}{ad}$; (*c*) 5, 5.

(5.) 72 inches; 36 inches.

(6.) 250.

MAY 1879.—(Page 213.)

(1.) Art. 47; Art. 49; $\dfrac{1}{3}a^{p}b^{-q}$ or $\dfrac{a^{p}}{3b^{q}}$.

(2.) 4.

(3.) $\dfrac{x^2+x-12}{x^2-x-12}$.

(4.) (*a*) -3; (*b*) $4\frac{1}{2}$; (*c*) $\dfrac{(b+a)c}{a^2+b^2}$, $\dfrac{(b-a)c}{a^2+b^2}$.

(5.) £17 for an ox; £26 for a cow.

(6.) £175; £225.

MAY 1880.—(Page 213.)

(1.) $-\dfrac{1}{7}$.

(2.) $a^3+2a^2x+2ax^2+x^3$.

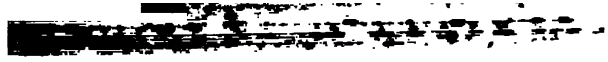
(3.) $(a-b)(a-c)(b-c)$; -30.

(4.) (*a*) $\frac{3}{4}$; (*b*) 2, -1.

(5.) £140.

(6.) $\dfrac{10}{(x-5)(x^2+2x-5)(x+5)}$.

PART II.



ALGEBRA.

PART SECOND.

CHAPTER XI.

INVOLUTION AND EVOLUTION.

8. **Involution** is the process of raising a quantity to a
er power, or of multiplying it one or more times by
; and it is indicated by placing the quantity to be
ted on within brackets, with the required power
de as an index: thus, $(a^2)^3$ denotes the third power
, and $(a-x)^4$ the fourth power of $a-x$.
e have already repeatedly had occasion to raise quan-
to higher powers, and the subject is here formally
up only for the purpose of pointing out one or two
ral rules that are applicable to it, and some short
ods of performing operations under it.

9. Let it be required to raise a and $-a$ to the second,
, fourth, and fifth powers successively.
multiplication (Art. 50) we have—

$a \times a = a^2$, and $-a \times -a = a^2$, second power.
$a^2 \times a = a^3$, and $a^2 \times -a = -a^3$, third power.
$a^3 \times a = a^4$, and $-a^3 \times -a = a^4$, fourth power.
$a^4 \times a = a^5$, and $a^4 \times -a = -a^5$, fifth power.

130. Observe in these results that the odd powers of the negative quantity are negative and all the others are positive.

Raise a^3 to the fourth power, and $-x^2$ to the fifth.

By multiplication—

$(a^3)^4 = a^3 \times a^3 \times a^3 \times a^3 = a^{3+3+3+3} = a^{3\times 4} = a^{12}$, and

$(-x^2)^5 = -x^2 \times -x^2 \times -x^2 \times -x^2 \times -x^2 = -x^{2+2+2+2+2} = -x^{2\times 5} = -x^{10}$.

131. From consideration of the above and former results we derive the following

RULE FOR THE INVOLUTION OF SIMPLE QUANTITIES.—Prefix the proper sign (Art. 130), and multiply the index of the given quantity by the number indicating the power to which it is to be raised.

If the given quantity is a product, each factor must be raised to the required power; if a fraction, both numerator and denominator.

Illustrative Examples.

(1.) Raise x^2 to the third power.

$$(x^2)^3 = x^{2\times 3} = x^6.$$

(2.) Raise $-2y^3$ to the fourth power.

$$(-2y^3)^4 = 2^4 y^{3\times 4} = 16y^{12}.$$

(3.) Raise $3ab^3c^{-2}$ to the third power.

$$(3ab^3c^{-2})^3 = 27a^3b^9c^{-6}.$$

(4.) Raise $-\frac{a^4x^2y}{bc^3z^5}$ to the fifth power.

$$\left(-\frac{a^4x^2y}{bc^3z^5}\right)^5 = -\frac{a^{20}x^{10}y^5}{b^5c^{15}z^{25}}.$$

(5.) What is the nth power of $2a^3x^{-p}$?

$$(2a^3x^{-p})^n = 2^n a^{3n} x^{-pn}.$$

(6.) Find the nth power of $-ab^2c^{x+y}$.

$$(-ab^2c^{x+y})^n = (-1)^n a^n b^{2n} c^{nx+ny}.$$

The answer in this last question will be plus or minus according as n is even or odd.

132. An expression consisting of two terms is called a binomial; of three terms, a trinomial; of four or more, a multinomial or polynomial; and a single term is sometimes spoken of as a monomial.

133. Expansion of a Binomial.—By multiplication the binomial quantity $a+x$ may be raised to the second, third, fourth, and higher powers, with the following results:—

$(a+x)^2 = a^2 + 2ax + x^2.$

$(a+x)^3 = a^3 + 3a^2x + 3ax^2 + x^3.$

$(a+x)^4 = a^4 + 4a^3x + 6a^2x^2 + 4ax^3 + x^4.$

$(a+x)^5 = a^5 + 5a^4x + 10a^3x^2 + 10a^2x^3 + 5ax^4 + x^5.$

$(a+x)^6 = a^6 + 6a^5x + 15a^4x^2 + 20a^3x^3 + 15a^2x^4 + 6ax^5 + x^6.$

. . . . . .

In these expansions, observe—

First, that the number of terms is one greater than the power to which the binomial is to be raised. For instance, $(a+x)^3$ yields four terms, and $(a+x)^6$ yields seven.

Second, that the power of the first term is the *same* as that to which the binomial is raised; as in $(a+x)^5$ the first term is a^5.

Third, that the power of a decreases by unity in each succeeding term until it disappears, and that x, being first introduced in the second term, its powers regularly increase until in the last term it has an index equal to that of a in the first.

Fourth, that if the coefficient of any term be multiplied by the exponent of a in that term, and divided by the number indicating its position in the series, we shall obtain the coefficient of the succeeding term. Thus, in the expansion of $(a+x)^6$, if 15, the coefficient of the third term, be multiplied by 4, the exponent of a in that term, and divided by 3, we shall have $\frac{15 \times 4}{3} = 20$, the coefficient of

the fourth term. Similarly, the coefficient of the fifth term will be obtained from the fourth; thus, $\frac{20 \times 3}{4} = 15$.

Fifth, that the coefficient of the second term is always the same as the exponent of the first. This is indeed only a particular case of the last observation, for in the expansion of $(a+x)^6$, the coefficient of the second term is $\frac{1 \times 6}{1} = 6$.

134. If the binomial $a-x$ be raised by multiplication to the second, third, and higher powers, we shall have—

$(a-x)^2 = a^2 - 2ax + x^2$.

$(a-x)^3 = a^3 - 3a^2x + 3ax^2 - x^3$.

$(a-x)^4 = a^4 - 4a^3x + 6a^2x^2 - 4ax^3 + x^4$.

$(a-x)^5 = a^5 - 5a^4x + 10a^3x^2 - 10a^2x^3 + 5ax^4 - x^5$.

$(a-x)^6 = a^6 - 6a^5x + 15a^4x^2 - 20a^3x^3 + 15a^2x^4 - 6ax^5 + x^6$.

. . . . . .

Here it will be observed that the terms are alternately plus and minus, and that in all other respects the expansions of the several powers of $a-x$ are the same as those of $a+x$.

Assuming, for the present, that these results are always true, we may, by their aid, expand any binomial, without having recourse to the slow method of multiplication.

Illustrative Examples.

(1.) Raise $a-b$ to the seventh power.

$(a-b)^7 = a^7 - 7a^6b + 21a^5b^2 - 35a^4b^3 + 35a^3b^4 - 21a^2b^5 + 7ab^6 - b^7$.

The only thing here calling for special remark is the manner of finding the several coefficients. That of the second term is the same as the exponent in the first; that of the third is obtained by multiplying the coefficient of the second by the exponent of a in it, and dividing by 2,

thus, $\frac{7 \times 6}{2} = 21$; the coefficient of the fourth term is similarly derived from that of the third, thus, $\frac{21 \times 5}{3} = 35$; and so on for the others.

(2.) Expand $(a^2+4)^4$.

$$(a^2+4)^4 = (a^2)^4 + 4(a^2)^3(4) + 6(a^2)^2(4)^2 + 4(a^2)(4)^3 + (4)^4$$
$$= a^8 + 16a^6 + 96a^4 + 256a^2 + 256.$$

(3.) Find the fifth power of $1-3x$.

$$(1-3x)^5 = (1)^5 - 5(1)^4(3x) + 10(1)^3(3x)^2 - 10(1)^2(3x)^3 + 5(1)(3x)^4 - (3x)^5.$$
$$= 1 - 15x + 90x^2 - 270x^3 + 405x^4 - 243x^5.$$

(4.) What is the third power of $\frac{2}{a} + \frac{5a}{x}$?

$$\left(\frac{2}{a} + \frac{5a}{x}\right)^3 = \left(\frac{2x+5a^2}{ax}\right)^3 = \frac{1}{a^3x^3}(2x+5a^2)^3$$
$$= \frac{1}{a^3x^3}\{(2x)^3 + 3(2x)^2(5a^2) + 3(2x)(5a^2)^2 + (5a^2)^3\}$$
$$= \frac{1}{a^3x^3}(8x^3 + 60a^2x^2 + 150a^4x + 125a^6)$$
$$= \frac{8}{a^3} + \frac{60}{ax} + \frac{150a}{x^2} + \frac{125a^3}{x^3}.$$

This example might, of course, have been wrought without reducing to the least common denominator.

135. The above method of expanding any power of a binomial may be applied for a like purpose to trinomials and polynomials by writing them in the binomial form.

Illustrative Examples.

(1.) Expand $(a+b-2c)^3$.

$$(a+b-2c)^3 = \{(a+b)-2c\}^3$$
$$= (a+b)^3 - 3(a+b)^2(2c) + 3(a+b)(2c)^2 - (2c)^3$$
$$= a^3 + 3a^2b + 3ab^2 + b^3 - 6a^2c - 12abc - 6b^2c + 12ac^2 + 12bc^2 - 8c^3.$$

(2.) Find the second power of $a+b+c+d$.

$$\{(a+b)+(c+d)\}^2=(a+b)^2+2(a+b)(c+d)+(c+d)^2$$
$$=a^2+2ab+b^2+2ac+2ad+2bc+2bd+c^2+2cd+d^2.$$

In this result we have the square of each of the terms, and twice the product of every possible pair of them.

It may also be written thus—

$$a^2+2a(b+c+d)+b^2+2b(c+d)+c^2+2cd+d^2.$$

From which it appears that the square of a quantity of four terms is equal to the square of every term increased by twice the product of each into the sum of all those that come after it.

This also holds true for the square of a quantity of any number of terms.

Illustrative Example.

Expand $(1-2x+3x^2-2x^3+x^4)^2$.

Writing in the first line the square of the first term, and twice the first multiplied by every succeeding term; in the second line, the square of the second term, and twice the second by all that come after it; and so on to the end, we have—

$$\begin{array}{rrrrrrrr} 1-4x+ & 6x^2- & 4x^3+ & 2x^4 & & & & \\ & 4x^2- & 12x^3+ & 8x^4- & 4x^5 & & & \\ & & & 9x^4- & 12x^5+ & 6x^6 & & \\ & & & & & 4x^6- & 4x^7 & \\ & & & & & & +x^8 & \\ \hline 1-4x+ & 10x^2- & 16x^3+ & 19x^4- & 16x^5+ & 10x^6- & 4x^7+ & x^8. \end{array}$$

EXAMPLES FOR PRACTICE.—XLVI.

(1.) Express the third powers of $2ax^3$, $-4bc^2y^4$, $\frac{a^2c^3x^3y^2}{mnz}$, and $-5x^n$.

(2.) What are the fourth powers of a^3x, $3ax^{-2}$, and $\frac{2}{3}\cdot\frac{c^2}{y^3}$?

(3.) Find the sixth powers of $a^m y^{-n}$, $\frac{2a^4}{x^{-1}}$, and $-z^3$.

(4.) Raise $2a^2$, $-x^m y^n$, and x^{p+q} to the mth power.

(5.) Expand $(x+y)^6$ and $(a-2x)^5$.

(6.) Simplify $\{(a^m)^n\}^p$ and $\{(x^{-m})^{-n}\}^{-p}$.

(7.) Raise $1+4x$ to the fourth power.

(8.) Find the third power of x^2-x+2.

(9.) Expand $(3a^2+5)^5$ and $(3a^2-5)^5$.

(10.) What is the eighth power of $x+a$?

(11.) Simplify $(a+2b)^7+(a-2b)^7$.

(12.) Square x^3-2x^2-2x+1.

(13.) Expand $(x^2+x+1)^3(x^2-x+1)^3$.

(14.) Find the sixth power of $a-\frac{2}{3}$.

(15.) Simplify $(x^3-1)^4-(1-x^3)^3$.

(16.) Reduce to its simplest form—

$$(ax-by+cz)^2-1-(ax-by+cz-1)^2.$$

(17.) Expand $(x-2+x^{-1})^3$.

(18.) Find the coefficient of x^6 in the expansion of $(x-\frac{3}{2})^9$ and in that of $(x^2-3x+1)^4(x^2-1)$.

136. Evolution is the inverse of Involution; or, it is the process of finding that quantity which, on being involved to a required power, will yield a given quantity. The quantity so found is called the root of the given one. Thus, a^2 raised to the third power becomes a^6; a^2 is, therefore, the third root of a^6. Similarly, $a-x$ is the fourth root of $(a-x)^4$.

As explained in Art. 123, the signs $\surd$, $\sqrt[3]{}$, $\sqrt[4]{}$ are employed to indicate respectively the second, third, and fourth roots of the quantities to which they are prefixed.

137. In Art. 130 it was found that all the powers of a positive quantity and the even powers of a negative one were positive, while the odd powers of a negative one

were negative. From this it follows that the odd root of any quantity will have the same sign as the quantity itself; that the even root of a positive quantity will be either positive or negative; and that a negative quantity cannot have an even root.

138. A quantity which has not an exact root is called an irrational quantity or a surd (Art. 124), and the even root of a negative quantity is spoken of as an impossible or imaginary quantity.

139. Roots of Simple Quantities.—As a simple quantity is raised to a higher power by multiplying its index (Art. 131), the root of a simple quantity will be found by dividing the index of the given quantity by the number indicating the root required, and prefixing the proper sign (Art. 137). If the given quantity is a product, the index of each factor must be divided; and if a fraction, the indices of numerator and denominator. A numerical coefficient must have its root extracted by the rules of arithmetic.

Illustrative Examples.

(1.) Find the fourth root of $16a^4x^8$.

$$\sqrt[4]{16a^4x^8} = \sqrt[4]{2^4a^4x^8} = 2ax^2 \text{ or } -2ax^2.$$

The uncertainty as to the proper sign is generally expressed by prefixing $\pm$ (read *plus or minus*) to the root. The above would then stand thus:—

$$\sqrt[4]{16a^4x^8} = \pm 2ax^2.$$

In many cases, however, we may be satisfied with writing only the positive result, the negative one being understood.

(2.) What is the fifth root of $-\frac{243x^{10}}{3125y^{15}}$?

$$\sqrt[5]{-\frac{243x^{10}}{3125y^{15}}} = \sqrt[5]{-\frac{3^5x^{10}}{5^5y^{15}}} = -\frac{3x^2}{5y^3}.$$

(3.) Find the mth root of $a^mb^{-2m}y^{mn}$.

$$\sqrt[m]{a^mb^{-2m}y^{mn}} = ab^{-2}y^n.$$

The sign of this answer will be plus if m be odd, either plus or minus if m be even.

EXAMPLES FOR PRACTICE.—XLVII.

(1.) Find the square root of $144x^4$.

(2.) What is the third root of $-125x^9y^6z^3$?

(3.) Extract the fourth root of $81\frac{a^4x^8}{b^8y^4}$.

(4.) Find the fifth root of $\frac{32x^{10}y^{15}}{243a^5z^{20}}$.

(5.) Express the third root of $a^6(x-y)^3$.

(6.) What is the sixth root of $(a-b)^6(x^2-y^2)^{12}$?

Find the roots of the following quantities, as indicated by the root-sign prefixed:—

(7.) $\sqrt[5]{x^5y^{10}}$ and $\sqrt[6]{a^{6n}b^{18m}}$.

(8.) $\sqrt{\sqrt[3]{a^{12}x^6}}$ and $\sqrt[3]{\sqrt[5]{-x^{15}y^{45}}}$.

(9.) $\sqrt[5]{\{5^{15}(a-x)^{10}(b-x)^{-5}\}}$.

(10.) $\sqrt[6]{\{(a+x)^2(a-x)^2\}^3}$.

(11.) $\sqrt[4]{a^{5n+1}\times a^{3n-5}}$.

(12.) $\sqrt[n]{\frac{x^{n^2}\{(b-y)^n\}^m}{\{(a-x)^p\}^nz^{nq+n}}}$.

140. Square Root of a Compound Quantity.—The square

of $a+b$ is $a^2+2ab+b^2$, and therefore the square root of $a^2+2ab+b^2$ is $a+b$.

This being known, let us try to find a method of obtaining the square root of $a^2+2ab+b^2$ that will apply to similar quantities.

Observe that a, the first term of the root, is the square root of a^2, the first term of the given quantity, and that if a^2 be subtracted there will remain $2ab+b^2$. The work will stand thus—

$$\begin{array}{l} a^2+2ab+b^2 \,\big(\, a \\ \underline{a^2} \\ \quad 2ab+b^2 \end{array}$$

Knowing that b must be the next term of the root, we observe that it may be obtained by dividing the first term of this remainder by twice a; but in order that there may be no remainder when the product of divisor and quotient is taken from the dividend, the divisor will require to be $2a+b$. ($2a$ is spoken of as the *trial* divisor, and $2a+b$ as the *complete* divisor.)

The work will stand thus—

$$\begin{array}{rl} & a^2+2ab+b^2 \,\big(\, a+b \\ & \underline{a^2} \\ 2a+b\big) & \quad +2ab+b^2 \end{array}$$

If now $2a+b$ be multiplied by b, and the product subtracted from $2ab+b^2$, there will be no remainder.

The square of $a+b+c$ is $a^2+2ab+b^2+2ac+2bc+c^2$, and this may be thrown into the form $(a+b)^2+2(a+b)c+c^2$ (Art. 135). On extracting the square root of this quantity, it is obvious that after $a+b$ has been found, c may be obtained by using $2(a+b)$ as a divisor. But $a+b$ may be found precisely as in the previous example, so that the

work of extracting the square root of $a^2+2ab+b^2+2ac+2bc+c^2$ will stand thus—

$$\begin{array}{l|l|l} a & a^2+2ab+b^2+2ac+2bc+c^2 & a+b+c \\ a & a^2 & \\ \hline 2a+b & 2ab+b^2 & \\ b & 2ab+b^2 & \\ \hline 2a+2b+c & 2ac+2bc+c^2 & \\ & 2ac+2bc+c^2 & \\ \hline \end{array}$$

141. From this we derive the following

RULE.—Arrange the terms according to the ascending or descending powers of one of the letters. Extract the square root of the first term. Place the quantity thus found in the positions usually occupied by the divisor and quotient in a sum of long division. Multiply the divisor and quotient together, and subtract the product from the given quantity. Divide the remainder by twice the part of the root found, as a trial divisor: this gives the second term of the root. Annex this term to the trial divisor to form the complete divisor. Multiply the complete divisor by the term of the root last found. Subtract the product. If there is still a remainder, use twice the part of the root obtained as a trial divisor, and proceed as before.

Illustrative Examples.

(1.) Find the square root of $4a^2+12ab+9b^2$.

$$\begin{array}{l|l|l} 2a & 4a^2+12ab+9b^2 & 2a+3b, \textit{ Ans.} \\ 2a & 4a^2 & \\ \hline 4a+3b & 12ab+9b^2 & \\ & 12ab+9b^2 & \\ \hline \end{array}$$

(2.) Find the square root of

$$6ax + 9x^2 - 24xy + a^2 - 8ay + 16y^2.$$

Arrange according to descending powers of a.

$$\begin{array}{l|l|l}
a & a^2 + 6ax + 9x^2 - 8ay - 24xy + 16y^2 & a + 3x - 4y \\
a & a^2 & \\
\hline
2a + 3x & 6ax + 9x^2 & \\
\text{A}\quad 3x & 6ax + 9x^2 & \\
\hline
2a + 6x - 4y & - 8ay - 24xy + 16y^2 & \\
 & - 8ay - 24xy + 16y^2 &
\end{array}$$

The square root is therefore $a + 3x - 4y$.

At A, $3x$ has been added in order to double the part of the root found, and so form the second trial divisor.

(3.) Extract the square root of

$$9x^4 - 12x^3 + 10x^2 - 34x + 21 - \frac{10}{x} + \frac{25}{x^2}.$$

$$\begin{array}{l|l|l|l}
3x^2 & 9x^4 - 12x^3 + 10x^2 - 34x + 21 - \frac{10}{x} + \frac{25}{x^2} & & 3x^2 - 2x \\
3x^2 & 9x^4 & & + 1 - \frac{5}{x}. \\
\hline
6x^2 - 2x & - 12x^3 + 10x^2 & & \\
 - 2x & - 12x^3 + 4x^2 & & \\
\hline
6x^2 - 4x + 1 & & 6x^2 - 34x + 21 & \\
 + 1 & & 6x^2 - 4x + 1 & \\
\hline
6x^2 - 4x + 2 - \frac{5}{x} & & - 30x + 20 - \frac{10}{x} + \frac{25}{x^2} & \\
 & & - 30x + 20 - \frac{10}{x} + \frac{25}{x^2} &
\end{array}$$

The required root is therefore $3x^2 - 2x + 1 - \frac{5}{x}$.

When the given quantity has no square root, an approximation to it may be found by continuing the above

process until what is considered a sufficient number of terms has been obtained.

(4.) Find approximately the square root of

$$a^2 - ax + x^2.$$

$$\begin{array}{l|l}
a & a^2 - ax + x^2 \left(a - \frac{x}{2} + \frac{3x^2}{8a} + \frac{3x^3}{16a^2} + \frac{3x^4}{128a^3} - \text{etc.}\right. \\
\underline{a} & \underline{a^2} \\
2a - \frac{x}{2} & -ax + x^2 \\
\underline{-\frac{x}{2}} & \underline{-ax + \frac{x^2}{4}} \\
2a - x + \frac{3x^2}{8a} & \frac{3x^2}{4} \\
\underline{+\frac{3x^2}{8a}} & \underline{\frac{3x^2}{4} - \frac{3x^3}{8a} + \frac{9x^4}{64a^2}} \\
2a - x + \frac{3x^2}{4a} + \frac{3x^3}{16a^2} & \frac{3x^3}{8a} - \frac{9x^4}{64a^2} \\
\underline{+\frac{3x^3}{16a^2}} & \underline{\frac{3x^3}{8a} - \frac{3x^4}{16a^2} + \frac{9x^5}{64a^3} + \frac{9x^6}{16^2a^4}} \\
2a - x + \frac{3x^2}{4a} + \frac{3x^3}{8a^2} & \frac{3x^4}{64a^2} - \frac{9x^5}{64a^3} - \frac{9x^6}{16^2a^4}
\end{array}$$

The approximate square root is therefore

$$a - \frac{x}{2} + \frac{3x^2}{8a} + \frac{3x^3}{16a^2} + \frac{3x^4}{128a^3} - \text{etc.}$$

EXAMPLES FOR PRACTICE.—XLVIII.

Find the square roots of the following quantities :—

(1.) $a^2 - 6ax + 9x^2$.

(2.) $16x^2 + 8x + 1$.

(3.) $9a^4 - 12a^3b + 10a^2b^2 - 4ab^3 + b^4$.

(4.) $4x^2+16x+36+\frac{40}{x}+\frac{25}{x^2}$.

(5.) $\frac{x^6}{9}-\frac{x^3y^2z}{3}+\frac{y^4z^2}{4}$.

(6.) $\frac{4}{9}x^2+\frac{9}{16}-x$.

(7.) $x^4+a^4-4ax(x^2+a^2)+6a^2x^2$.

(8.) $x^4-x^3+\frac{9}{4}x^2-2x+\frac{3}{2}-x^{-1}+\frac{1}{4}x^{-2}$.

(9.) $\frac{a^2x^2}{y^2}-\frac{2abx}{z}+\frac{2acz}{y}+\frac{b^2y^2}{z^2}-\frac{2bcy}{x}+\frac{c^2z^2}{x^2}$.

(10.) $a^{4m}-2a^{3m}b+5a^{2m}b^2-4a^mb^3+4b^4$.

(11.) $1-x$ and a^2+4x^2 to four terms.

(12.) $\frac{1}{1+x}$ and $\frac{a+x}{a-x}$ to five terms.

142. Cube Root.—The usual method of finding the cube root of a compound quantity is derived from the expression for the third power of a binomial.

$(a+b)^3=a^3+3a^2b+3ab^2+b^3$, and therefore the cube root of $a^3+3a^2b+3ab^2+b^3$ is $a+b$.

$$\begin{array}{rl|l} & a^3+3a^2b+3ab^2+b^3 & a+b \\ & a^3 & \\ \hline 3a^2+3ab+b^2 & 3a^2b+3ab^2+b^3 & =(3a^2+3ab+b^2)b \\ & 3a^2b+3ab^2+b^3 & \end{array}$$

Observe that a, the first term of the required root, is the cube root of the first term of the given quantity, and that on a^3 being subtracted, the remainder, $3a^2b+3ab^2+b^3$, or $(3a^2+3ab+b^2)b$, yields b, the second term of the root, on being divided by $3a^2+3ab+b^2$.

If it be required to find the cube root of $a^3+3a^2b+3ab^2+b^3+3a^2c+6abc+3b^2c+3ac^2+3bc^2+c^3$, which is the third power of the trinomial $a+b+c$, we find, by throwing it into the form of an expanded binomial, that the method employed in the above example will also suffice for this.

$$
\begin{array}{lll}
 & (a+b)^3+3(a+b)^2c+3(a+b)c^2+c^3 & \left(\begin{array}{l} a+b \\ +c \end{array}\right. \\
 & (a+b)^3 & \\
\hline
3(a+b)^2+3(a+b)c+c^2 & 3(a+b)^2c+3(a+b)c^2+c^3 & \\
 & 3(a+b)^2c+3(a+b)c^2+c^3 &
\end{array}
$$

143. From these examples we have the following

RULE.—Arrange the terms according to the ascending or descending powers of one of the letters. Extract the cube root of the first: this will give the first term of the root. Subtract the third power of this term from the given quantity. Divide the remainder by three times the square of the part of the root found: this is called the trial divisor, and gives the second term of the root. To the trial divisor add three times the product of the first and second terms of the root, and the square of the second: this forms the complete divisor. Multiply the complete divisor by the term last found, and subtract the product. If there be still a remainder, divide it by three times the square of the part of the root found: this will give the third term. To the trial divisor here used add three times the product of the first two terms by the third, and the square of the third. Multiply and subtract as before.

Proceed according to the above method until there is no remainder, or until a sufficient number of terms have been found.

Illustrative Examples.

(1.) Find the cube root of $64a^3 - 144a^2b + 108ab^2 - 27b^3$.

$$
\begin{array}{lll|l}
3(4a)^2 & = 48a^2 & & 64a^3 - 144a^2b + 108ab^2 - 27b^3 \;(\; 4a - 3b \\
3(4a)(-3b) & = \quad -36ab & & 64a^3 \\
(-3b)^2 & = \quad +9b^2 & & \hline
 & 48a^2 - 36ab + 9b^2 & & -144a^2b + 108ab^2 - 27b^3 \\
 & & & -144a^2b + 108ab^2 - 27b^3
\end{array}
$$

(2.) Find the cube root of $125x^6 - 225x^5 + 210x^4 - 117x^3 + 42x^2 - 9x + 1$.

$$
\begin{array}{lll|l}
3(5x^2)^2 & = 75x^4 & & 125x^6 - 225x^5 + 210x^4 - 117x^3 + 42x^2 - 9x + 1 \;(\; 5x^2 - 3x + 1 \\
3(5x^2)(-3x) & = \quad -45x^3 & & 125x^6 \\
(-3x)^2 & = \quad +9x^2 & & \hline
 & 75x^4 - 45x^3 + 9x^2 & & -225x^5 + 210x^4 - 117x^3 \\
 & & & -225x^5 + 135x^4 - 27x^3 \\
 & & & \hline
 & & & 75x^4 - 90x^3 + 42x^2 - 9x + 1 \\
3(5x^2 - 3x)^2 & = 75x^4 - 90x^3 + 27x^2 \quad \text{(A)} & & 75x^4 - 90x^3 + 42x^2 - 9x + 1 \\
3(5x^2 - 3x) \times 1 & = \quad 15x^2 - 9x & & \\
(1)^2 & = \quad +1 & & \\
 & 75x^4 - 90x^3 + 42x^2 - 9x + 1 & &
\end{array}
$$

Note.—The trial divisor (A) may be readily obtained by bringing down from the line above it—*once* the first term, *twice* the second, and *thrice* the third; or by taking the sum of the preceding three lines with the last term of the third doubled.

EXAMPLES FOR PRACTICE.—XLIX.

Extract the cube roots of the following quantities :—

(1.) $x^3+6x^2+12x+8$.

(2.) $64x^6-48x^4y+12x^2y^2-y^3$.

(3.) $27a^3-54a^2b+36ab^2-8b^3+27a^2c-36abc+12b^2c+9ac^2-6bc^2+c^3$.

(4.) $8x^6-12x^5y+42x^4y^2-37x^3y^3+63x^2y^4-27xy^5+27y^6$.

(5.) $x^9+3x^8+6x^7+10x^6+12x^5+12x^4+10x^3+6x^2+3x+1$.

(6.) $a^3-6a^2+15a-20+\frac{15}{a}-\frac{6}{a^2}+\frac{1}{a^3}$.

(7.) $1+\frac{3}{2}xy-\frac{1}{4}x^2y^2-\frac{7}{8}x^3y^3+\frac{1}{12}x^4y^4+\frac{1}{6}x^5y^5-\frac{1}{27}x^6y^6$.

(8.) $125x^6-75x^4y^n+15x^2y^{2n}-y^{3n}$.

(9.) $8x^{12}-12x^{11}+12x^{10}-7x^9+3x^8-\frac{3}{4}x^7+\frac{1}{8}x^6$.

(10.) $a^3x^3-3a^2bx^2+3(a^2c+ab^2)x-b^3-6abc+3(ac^2+b^2c)x^{-1}-3bc^2x^{-2}+c^3x^{-3}$.

(11.) $m^3-m^2n+mn^2-n^3$ to four terms.

(12.) $1-x$ to five terms.

144. General Rule for Extraction of Roots.—By the help of the two previous rules several other roots can be found; as, for instance, the fourth root may be obtained by taking the square root of the square root, and the sixth by taking the square root of the cube root. They may also be found by a method which is applicable to all roots. It will be best explained by an example.

Required the fourth root of

$81x^8-216x^7+324x^6-312x^5+214x^4-104x^3+36x^2-8x+1$.

(1.)	(2.)	(3.)	(4.)
$3x^2$	$9x^4$	$27x^6$	$81x^8-216x^7+324x^6-312x^5+214x^4-104x^3+36x^2-8x+1$
$3x^2$	$18x^4$	$81x^6$	$81x^8$
$6x^2$	$27x^4$	$108x^6-108x^5+48x^4-8x^3$	$-216x^7+324x^6-312x^5+214x^4$
$3x^2$	$27x^4$	$-108x^5+96x^4-24x^3$	$-216x^7+216x^6-96x^5+16x^4$
$9x^2$	$54x^4-24x^3+4x^2$	$108x^6-216x^5+144x^4-32x^3$	$108x^6-216x^5+198x^4-104x^3+36x^2-8x+1$
$3x^2$	$-24x^3+8x^2$	$54x^4-72x^3+36x^2-8x+1$	$108x^6-216x^5+198x^4-104x^3+36x^2-8x+1$
$12x^2-2x$	$54x^4-48x^3+12x^2$	$108x^6-216x^5+198x^4-104x^3+36x^2-8x+1$	
$-2x$	$-24x^3+12x^2$		
$12x^2-4x$	$54x^4-72x^3+24x^2$		
$-2x$	$12x^2-8x+1$		
$12x^2-6x$	$54x^4-72x^3+36x^2-8x+1$		
$-2x$			
$12x^2-8x+1$			

Ans. $= 3x^2 - 2x + 1.$

For the fourth root, the work is arranged in four columns, which are here numbered (1.), (2.), (3.), (4.).

The quantity whose root is to be found having been placed under (4.), the fourth root, $3x^2$, of its first term is taken to form the first term of the answer.

This term, $3x^2$, is then placed in column (1.); its second power, $9x^4$, in column (2.); its third power, $27x^6$, in column (3.); and its fourth power, $81x^8$, under the first term of column (4.), from which it is subtracted, the whole or a portion of the remaining terms being brought down.

Next, $3x^2$ is placed a second time in column (1.), and added to the term already there. The

sum of these terms, $6x^2$, is multiplied by $3x^2$, and the product, $18x^4$, is placed in column (2.), and added to the term already there.

The sum, $27x^4$, is multiplied by $3x^2$, and the product, $81x^6$, is placed in column (3.), and added to the quantity already standing there, the sum being $108x^6$.

The term $3x^2$ is now placed a third time in column (1.), and added to what stands there.

The sum, $9x^2$, is multiplied by $3x^2$, and the product, $27x^4$, is added to column (2.), the sum being $54x^4$.

$3x^2$ is for the fourth time placed in column (1.), and added, giving the sum $12x^2$. This completes the first part of the process.

The second term of the answer, $-2x$, is obtained by dividing the remainder standing in column (4.) by the sum in column (3.).

$-2x$ is then annexed to column (1.).

The sum, $12x^2-2x$, is multiplied by $-2x$, and the product, $-24x^3+4x^2$, is annexed to column (2.).

The sum, $54x^4-24x^3+4x^2$, is, in its turn, multiplied by $-2x$, and the product, $-108x^5+48x^4-8x^3$, is annexed to column (3.).

Similarly, the product, $-216x^7+216x^6-96x^5+16x^4$, is obtained and placed in column (4.), from which it is subtracted.

$-2x$ is now added a second time to column (1.), and the alternate addition and multiplication proceed as above until the third term is reached.

$-2x$ is added a third time to column (1.), and the product as before is added to column (2.).

$-2x$ is added a fourth time to column (1.).

The fourth column is then divided by the third to find the next term of the answer: it is $+1$.

$+1$ being added to column (1.), and the process shown above repeated, it is found that there is no remainder in the fourth column.

$\therefore 3x^2 - 2x + 1$ is the required root.

The method adopted in the above example may be applied to the extraction of any root by employing a greater or less number of columns according to the root required. For the third root, three columns; for the fifth root, five; and so on: the quantity whose root is to be found being always placed in the last column.

EXAMPLES FOR PRACTICE.—L.

(1.) Find the third root of $x^6 + 6x^5 - 40x^3 + 96x - 64$.

(2.) Find the fourth root of $a^4 - 2a^2x^2 + x^4$ to four terms.

(3.) Find the fifth root of $1 + x$ to five terms.

CHAPTER XII.

QUADRATIC EQUATIONS.

145. A quadratic equation is one that contains the second power, or the first and second powers, of the unknown quantity, but no higher one. It is also called an equation of the second degree, or of two dimensions.

When the second power alone appears in the equation, it is spoken of as a *pure* quadratic; but when both first and second powers enter into it, it is called a *compound* or *adfected* quadratic.

$3x^2 - 4 = 5 + 2x^2$ is an example of the former,

$4x^2 - 2x = 2x + 3$, of the latter.

146. Pure Quadratics.—In equations of this character the value of the second power of the unknown quantity may evidently be found in the same way as that of the unknown quantity itself in a simple equation.

Let the value of x be found from the equation

$$3x^2 - (5 - 2x^2) = (7 + x)(7 - x).$$

By previous rules

$$3x^2 - 5 + 2x^2 = 49 - x^2$$

And

$$6x^2 = 54.$$

$$\therefore\ x^2 = 9.$$

To find x, it will here be necessary to extract the square root of both sides of the equation; and as the even root of a positive quantity is either plus or minus (Art. 137), we shall then have $\pm x = \pm 3$.

Taking the upper signs, we get $+x = +3$; taking the lower, $-x = -3$, which is the same as $+x = +3$ (Art. 34): taking an upper and an under one, we have $+x = -3$ and $-x = +3$, this last being the same as $+x = -3$ (Art. 70). So that there are only two distinct results—namely, $x = +3$ and $x = -3$, and consequently it is unnecessary to write the double sign before x.

147. From this we derive the

RULE FOR THE SOLUTION OF PURE QUADRATICS.—Find, by the rules for the solution of Simple Equations, the value of the square of the unknown quantity, and extract its square root.

If the value found is not an exact square, its root can be found approximately, or it may be merely indicated, and left in the form of a surd (Art. 138).

Illustrative Examples.

(1.) Given $\dfrac{4x+3}{(3x-5)x} - \dfrac{4x-3}{(3x+5)x} = \dfrac{87}{7(x^2+3)}$, to find x.

Reducing the first side to L.C.D., we have—

$$\frac{12x^2+29x+15-(12x^2-29x+15)}{(9x^2-25)x} = \frac{87}{7(x^2+3)}$$

Collecting, $$\frac{58x}{(9x^2-25)x} = \frac{87}{7x^2+21}.$$

Cancelling, $$\frac{2}{9x^2-25} = \frac{3}{7x^2+21}.$$

$$\therefore\ 27x^2-75 = 14x^2+42.$$

And $$13x^2 = 117.$$

From which $$x^2 = 9, \text{ and } x = \pm 3.$$

.) Find the value of x from the equation

$$ax(x-b)+b^2=bx(x-a)+a^2.$$

$$ax^2-abx+b^2=bx^2-abx+a^2$$

$$(a-b)x^2=a^2-b^2$$

$$\therefore\ x^2=a+b, \text{ and } x=\pm\sqrt{a+b}, \text{ which}$$

the form of an irrational quantity.

.) Solve the equation $x-2=\dfrac{3x^2-x}{2x+3}$.

ring off fractions, $2x^2-x-6=3x^2-x$.

$$\therefore\ x^2=-6.$$

s a negative quantity has not an even root (Art. 138), rithmetical solution can be found, and the value of x only be expressed in the form of an impossible or ;inary quantity; thus, $x=\pm\sqrt{-6}$.

EXAMPLES FOR PRACTICE.—LI.

)lve the following equations:

(1.) $5x^2-5=3x^2+3.$

(2.) $x^2+1=2(23-2x^2).$

(3.) $\dfrac{3x^2-8}{8}=\dfrac{x^2-1}{3}.$

(4.) $\dfrac{7}{6x^2+1}=\dfrac{11}{12x^2-1}.$

(5.) $(x-2)^2=13-4x.$

(6.) $(3x+1)(x+2)=(x+3)(x+4).$

(7.) $\dfrac{2x-1}{3x+1}+\dfrac{2x+1}{3x-1}=\dfrac{10}{7}.$

(8.) $\dfrac{x^2-1}{x^2-4}-\dfrac{x^2-5}{x^2-8}=\dfrac{x^2-2}{x^2-5}-\dfrac{x^2-6}{x^2-9}.$

(9.) $a(x+b)^2=b(x+a)^2.$

(10.) $x^2+(3x-2)^2(4x+3)=4(x^2+2x-1)(9x+2).$

$$(11.)\quad \frac{2x-1}{x-1} - \frac{4(x+1)}{2x+1} = \frac{x-1}{x-2} - \frac{x+2}{x+1}.$$

$$(12.)\quad \frac{1-x+x^2}{1+x} + \frac{1+x+x^2}{1-x} = m.$$

148. Adfected Quadratics.—Equations containing both first and second powers of the unknown quantity, although named adfected or compound quadratics, are generally spoken of simply as quadratics or quadratic equations, and this practice will be maintained here. For their solution several methods may be employed.

First Method.

149. By resolution of the quadratic into factors.

Let x be given equal to 3, then $x-3=0$; and should x be also given equal to -5, then $x+5=0$.

If now these two expressions be multiplied together, we shall have $(x-3)(x+5)=0$; that is, $x^2+2x-15=0$, a quadratic equation, the product of two simple ones in which the same letter represents two different values.

Suppose that the quadratic equation $x^2+2x-15=0$ is given to us for solution. Plainly, if we could find the simple equations from which it was formed, we could tell from them the values of x.

Now these equations are readily got by resolving $x^2+2x-15=0$ into its factors (Art. 74). Doing this, we have $(x-3)(x+5)=0$. But if the product of two factors is nothing, then one or other of them must also equal nothing; and consequently, in this example, either $x-3=0$ or $x+5=0$. And $\therefore\ x=3$ or -5.

Again, if $x=\frac{3}{2}$ or $\frac{5}{6}$, we shall have the simple equations $x-\frac{3}{2}=0$, and $x-\frac{5}{6}=0$, from which we can easily form

the quadratic equation $x^2 - \frac{7}{3}x + \frac{5}{4} = 0$, or $12x^2 - 28x + 15 = 0$.

Suppose this equation given for solution. By trial, its factors are found to be $2x - 3$ and $6x - 5$; and as one at least of them must be equal to 0, we have either $2x = 3$ or $6x = 5$, and $\therefore\ x = \frac{3}{2}$ or $\frac{5}{6}$.

To enable the factoring to be more readily performed, and to render applicable the methods of Art. 75, it is advisable to divide the whole expression by the coefficient of x^2; thus—

$$x^2 - \frac{28}{12}x + \frac{15}{12} = 0,$$

and at the same time to multiply both lines of the third term by its denominator, as under—

$$x^2 - \frac{28}{12}x + \frac{180}{144} = 0.$$

Now applying Art. 75, we find that the two numbers which multiplied together make $\frac{180}{144}$, and added together make $-\frac{28}{12}$ are $-\frac{18}{12}$ and $-\frac{10}{12}$.

Therefore the above expression becomes

$$\left(x - \frac{18}{12}\right)\left(x - \frac{10}{12}\right) = 0 \text{ or } \left(x - \frac{3}{2}\right)\left(x - \frac{5}{6}\right) = 0.$$

From which $x = \frac{3}{2}$ or $\frac{5}{6}$.

Observe that 180 is the product of 12, the coefficient of x^2, and 15 the known quantity.

150. The process shown in this example suggests the following

RULE.—Collect like terms and arrange them all on one

side. Multiply the coefficient of x^2 and the known quantity together. Resolve this product into two factors whose algebraic sum is the coefficient of x. The numbers thus obtained, divided by the coefficient of x^2, and having their signs changed, are the required values of x.

Note.—When the equation cannot be resolved into factors, x has no rational value, and the method is not applicable.

Illustrative Examples.

(1.) Given $2x^2+x-6=0$, to find x.

Multiply the coefficient of x^2 by the known quantity.

This gives $2 \times -6 = -12$.

Find the factors of -12 whose sum is $+1$, the coefficient of x.

These factors are -3 and $+4$.

Change the signs of these, and divide them by 2, the coefficient of x^2.

$$\therefore\ x = \frac{3}{2} \text{ or } -\frac{4}{2} = 1\tfrac{1}{2} \text{ or } -2.$$

Proof.

$$2\left(\frac{3}{2}\right)^2 + \frac{3}{2} - 6 = \frac{9}{2} + \frac{3}{2} - 6 = 6 - 6 = 0.$$

And $$2(-2)^2 + (-2) - 6 = 8 - 8 = 0.$$

(2.) Given $(7x+3)(x-2)=(3x-5)(x+3)$, to find x.

Expanding, $7x^2-11x-6=3x^2+4x-15$.

Coll. and trans., $4x^2-15x+9=0$.

$$4 \times 9 = 36.$$

The factors of 36 whose sum is -15 are -12 and -3.

$$\therefore\ x = \frac{12}{4} \text{ or } \frac{3}{4}.$$

$$= 3 \text{ or } \frac{3}{4}.$$

Given $\dfrac{2x+5}{3x-4}+\dfrac{3x-8}{2x+1}=2$, to find x.

g away fractions—

$x^2+12x+5+9x^2-36x+32=2(6x^2-5x-4)$.

ıd trans., $x^2-14x+45=0$.

ıe coefficient of x^2 is *one*, we have merely to find the of $+45$ which together amount to -14.

These are -5 and -9.

$\therefore\ x=5$ or 9.

Given $(x+2)(2x-3)(3x-1)=(x+1)(4x-1)(5x-6)$, x.

ing, $6x^3+x^2-19x+6=20x^3-9x^2-23x+6$.

d trans., $14x^3-10x^2-4x=0$.

g by $2x$, $7x^2-5x-2=0$.

$7\times-2=-14=-7\times2$.

$\therefore\ x=\dfrac{7}{7}$ or $-\dfrac{2}{7}$.

$=1$ or $-\dfrac{2}{7}$.

ı the $2x$ divided out, we also get $x=0$, a third value; equation $14x^3-10x^2-4x=0$, on being reduced to becomes $2x(x-1)(7x+2)=0$; and as each factor equated to 0, we get the three values—

$$x=0,\ x=1,\ x=-\frac{2}{7}.$$

Given $mnx^2-(an-bm)x=ab$, to find x.

$mnx^2-(an-bm)x-ab=0$.

$mn\times-ab=-abmn=-an\times+bm$.

$\therefore\ x=\dfrac{an}{mn}$ or $-\dfrac{bm}{mn}$.

$=\dfrac{a}{m}$ or $-\dfrac{b}{n}$.

(6.) Given $6x^2 - 5x + 3 = 0$, to find x.

$$6 \times 3 = 18.$$

As no two factors of 18 have -5 for their sum, this expression cannot be resolved into factors, and x has no rational value.

EXAMPLES FOR PRACTICE.—LII.

Solve the following by the above method :—

(1.) $x^2 - 6x + 8 = 0$.

(2.) $x^2 + x = 20$.

(3.) $3x^2 + 7x = 6$.

(4.) $6(x^2 - 1) = 35x$.

(5.) $(2x - 1)^2 = 4x + 22$.

(6.) $(5x + 1)^2 + 4 = (4x + 1)^2 - 11x$.

(7.) $4(4x + 1)(x - 1) = (5x - 1)(3x - 1)$.

(8.) $\dfrac{3x+1}{4x+3} + \dfrac{x+1}{5x+1} = \dfrac{11}{12}$.

(9.) $\dfrac{x+3}{x+4} + \dfrac{x-3}{x+1} = \dfrac{4x-9}{2x-1}$.

(10.) $(x + 1)(x + 3)(x + 5) = (2x + 1)(x^2 + x + 15)$.

(11.) $abx^2 + a^2 - b^2 = (a^2 + b^2)x$.

(12.) $4x^2 - 7x = 3$.

SECOND METHOD.

151. By Completing Square.—Since $(x+a)^2 = x^2 + 2ax + a^2$, the expression $x^2 + 2ax$ may be converted into a square by adding to it a^2, which is the square of half the coefficient of x, and since $(x - a)^2 = x^2 - 2ax + a^2$, $x^2 - 2ax$ will also become a square by the addition of a^2.

If now we have for solution the equation $x^2 + 2ax = 3a^2$, by adding a^2 to the first side we will render it a complete square; but as the addition of a quantity to the one side

requires the addition of a like quantity to the other, the equation will become $x^2+2ax+a^2=a^2+3a^2$ or

$$(x+a)^2=4a^2.$$

Taking now the square root of each side, we get $\pm(x+a)=\pm 2a$, but, for the reason given in Art. 146, it is sufficient to write the double sign only on the second side; this gives $x+a=\pm 2a$, two simple equations, from which x may be determined.

$$\therefore \quad x=-a+2a \text{ or } -a-2a.$$
$$=a \text{ or } -3a.$$

If the given equation be of the form $ax^2+bx=c$, on dividing by the coefficient of x^2, we obtain $x^2+\frac{b}{a}x=\frac{c}{a}$, and completing the square of the first side as in last example, we have—

$$x^2+\frac{b}{a}x+\left(\frac{b}{2a}\right)^2=\frac{b^2}{4a^2}+\frac{c}{a}=\frac{b^2+4ac}{4a^2}.$$

If b^2+4ac be a square, say m^2, we have $x+\frac{b}{2a}=\pm\frac{m}{2a}$.

$$\text{And } \therefore \quad x=\frac{-b\pm m}{2a}.$$

But if b^2+4ac be not a square, its root can only be indicated, and x will then equal $\frac{-b\pm\sqrt{b^2+4ac}}{2a}$.

152. From consideration of the above, we obtain the following

RULE.—Arrange the unknown quantity and its square on the first side, and the known quantity on the other.

Let the square be positive, and stand first.

If necessary, multiply or divide all the terms of the equation by such a number as will make the coefficient of the first term unity.

Add to each side the square of half the coefficient of the unknown quantity.

Extract the square root of each side, and place the double sign before the second one.

From the two simple equations thus obtained, find the values of the unknown quantity.

Illustrative Examples.

(1.) Given $x^2 + 2x = 63$, to find x.

Completing square, $x^2 + 2x + 1 = 64$.

Extracting root, $x + 1 = \pm 8$.

$$\therefore\ x = -1 \pm 8 = 7 \text{ or } -9.$$

(2.) When $x^2 - 3x = 40$, what is the value of x?

The coefficient of x being here an odd number, its half is represented as the fraction $\frac{3}{2}$, the square of which is $\frac{9}{4}$; therefore completing square we have—

$$x^2 - 3x + \left(\frac{3}{2}\right)^2 = \frac{9}{4} + 40 = \frac{169}{4}.$$

Extracting root, $x - \frac{3}{2} = \pm \frac{13}{2}$.

$$\therefore\ x = \frac{3 \pm 13}{2} = 8 \text{ or } -5.$$

(3.) Given $8x - 5x^2 = -4$, to find x.

Rearranging so as to have $5x^2$ first, and positive—

$$5x^2 - 8x = 4.$$

Dividing by coefficient of x^2—

$$x^2 - \frac{8}{5}x = \frac{4}{5}.$$

Completing square—

$$x^2 - \frac{8}{5}x + \left(\frac{4}{5}\right)^2 = \frac{16}{25} + \frac{4}{5} = \frac{36}{25}.$$

Extracting root, $x - \frac{4}{5} = \pm\frac{6}{5}$.

$$\therefore\ x = \frac{4 \pm 6}{5} = 2 \text{ or } -\frac{2}{5}.$$

(4.) Solve the equation

$$(5x+4)^2 - (3x-1)^2 = (4x+2)^2 - (2x-1)^2.$$

By Theorem III., Art. 57—

$$(8x+3)(2x+5) = (6x+1)(2x+3).$$

Expanding, $16x^2 + 46x + 15 = 12x^2 + 20x + 3$.

Collecting, $4x^2 + 26x = -12$.

Dividing, $x^2 + \frac{13}{2}x = -3$.

Completing square,

$$x^2 + \frac{13}{2}x + \left(\frac{13}{4}\right)^2 = \frac{169}{16} - 3 = \frac{169-48}{16} = \frac{121}{16}.$$

Extracting root, $x + \frac{13}{4} = \pm\frac{11}{4}$.

$$\therefore\ x = \frac{-13 \pm 11}{4} = -\frac{1}{2} \text{ or } -6.$$

(5.) Find x from the following equation :—

$$\frac{x-3}{x-5} - \frac{x-1}{x-2} = \frac{x-4}{x-6} - \frac{x-2}{x-3}.$$

Reducing each side to a common denominator,

$$\frac{x^2-5x+6-x^2+6x-5}{x^2-7x+10} = \frac{x^2-7x+12-x^2+8x-12}{x^2-9x+18}.$$

Collecting, $\frac{x+1}{x^2-7x+10} = \frac{x}{x^2-9x+18} = \frac{1}{2x-8}$ (Art. 109).

Clearing fractions, $2x^2 - 8x = x^2 - 9x + 18$.

$$\therefore\ x^2 + x = 18.$$

Compl. square, $x^2 + x + \frac{1}{4} = 18\frac{1}{4} = \frac{73}{4}$.

Extracting root, $x+\frac{1}{2}=\pm\frac{\sqrt{73}}{2}$.

$$\therefore\ x=\frac{-1\pm\sqrt{73}}{2}.$$

(6.) Given $\frac{x^2+1}{2}=\frac{a^2+b^2}{a^2-b^2}x$, to find x.

Simplifying and transposing,

$$x^2-\frac{2(a^2+b^2)}{a^2-b^2}x=-1.$$

Completing square,

$$x^2-\frac{2(a^2+b^2)}{a^2-b^2}x+\left(\frac{a^2+b^2}{a^2-b^2}\right)^2=\frac{(a^2+b^2)^2-(a^2-b^2)^2}{(a^2-b^2)^2}=\frac{4a^2b^2}{(a^2-b^2)^2}.$$

Extracting root, $x-\frac{a^2+b^2}{a^2-b^2}=\frac{\pm 2ab}{a^2-b^2}$.

$$\therefore\ x=\frac{a^2\pm 2ab+b^2}{a^2-b^2}=\frac{(a+b)^2}{a^2-b^2}\text{ or }\frac{(a-b)^2}{a^2-b^2}$$

$$=\frac{a+b}{a-b}\text{ or }\frac{a-b}{a+b}.$$

EXAMPLES FOR PRACTICE.—LIII.

Find the value of x in each of the following equations:—

(1.) $x^2-4x=5$.
(2.) $x^2+x=7x-5$.
(3.) $3(x^2-x)-6(x-2)=2(x^2-1)$.
(4.) $6(x+6)-x(x+1)=0$.
(5.) $(x-2)^2=2x$.
(6.) $2x^2-11x=40$.
(7.) $(x-4)^2+(2x-5)^2=(3x-7)^2$.
(8.) $(4x-3)(3x-4)-(x-4)(3x-1)=4$.

(9.) $\frac{x-1}{x}+\frac{x}{x-1}=1.$

(10.) $3{\cdot}75x^2+1=4x.$

(11.) $\frac{3x-2}{x+4}-\frac{x-1}{x+5}=3.$

(12.) $ax^2+(a-1)x=1.$

(13.) $\frac{x}{2}+\frac{2}{x}=\frac{x}{5}-\frac{5}{3}.$

(14.) $\frac{4-x}{4+x}+\frac{3+x}{3-x}=\frac{12-2x}{7+x}.$

(15.) $bx^2=\frac{a^2x-ab+b^2}{a+b}.$

(16.) $\frac{(x+1)(x-3)}{(x+2)(x+3)}=\frac{(x-2)(x+3)}{(x-4)(x-3)}.$

(17.) $\frac{1}{(x-1)(x-3)}+\frac{1}{(a+1)(a+3)}=\frac{1}{(a+1)(x-1)}+\frac{1}{(a+3)(x-3)}.$

(18.) $\frac{1}{x+a}+\frac{1}{x+b}+\frac{1}{x+c}=\frac{3}{x}.$

Third Method.

153. By rendering Coefficient of x^2 a Square.—In order to avoid fractions, it is sometimes recommended to multiply the whole of the given equation by four times the coefficient of x^2, and then to add to both sides the square of the coefficient of x. The root of both sides being extracted, the value of x may be found from the resulting equation.

Illustrative Examples.

Given $6x^2+11x=10$, to find x.

Multiplying by 4×6, or 24, we have—

$$144x^2+264x=240.$$

Adding the square of 11 to both sides,

$$144x^2 + 264x + 121 = 240 + 121 = 361.$$

Extracting root of both sides,

$$12x + 11 = \pm 19.$$

From which, $12x = -11 \pm 19 = 8$ or -30.

And $\therefore\ x = \frac{2}{3}$ or $-\frac{5}{2}$.

(2.) Find x from the equation $15x^2 - 43x = -13$.

Multiplying by 60, $900x^2 - 2580x = -780$.

Compl. square, $900x^2 - 2580x + (43)^2 = 1849 - 780 = 1069$.

Extracting root, $30x - 43 = \pm \sqrt{1069}$.

$$30x = 43 \pm \sqrt{1069}.$$

$$\therefore\ x = \frac{1}{30}(43 \pm \sqrt{1069}).$$

EXAMPLES FOR PRACTICE.—LIV.

Find the value of x in the following equations:—

(1.) $8x^2 + 45x = 18$.
(2.) $13x^2 + 33x = 18$.
(3.) $20x^2 - 31x + 12 = 0$.
(4.) $11x^2 - 5x = 1$.
(5.) $32x^2 - 51x + 23 = 0$.
(6.) $abx^2 + acx = bc$.

154. Equations that may be solved like Quadratics.—Equations not really quadratics may frequently be solved by being thrown into the quadratic form.

The following examples will illustrate the methods that may be adopted:—

(1.) Given $x^4 - 13x^2 + 36 = 0$, to find x.

If y be put for x^2, then y^2 will stand for x^4, and the above equation may be written $y^2 - 13y + 36 = 0$, a quadratic which on solution gives $y = 4$ or 9, that is, $x^2 = 4$ or 9.

And $\therefore\ x = \pm 2$ or ± 3.

Observe that there are here four different values of the

unknown quantity, and that the fourth power of this quantity appears in the given equation.

(2.) Given $3x + \sqrt{x} = 2$, to find x.

Let $\sqrt{x} = y$, then $x = y^2$.

And the equation becomes $3y^2 + y = 2$, a quadratic.

Solving, we find y, that is, $\sqrt{x} = \frac{2}{3}$ or -1.

From which $x = \frac{4}{9}$ or 1.

The second value of x, however, will not satisfy the given equation unless $\sqrt{x}$ be taken with a minus sign; thus, $3x - \sqrt{x} = 2$, in which case the first value of x will not apply.

(3.) Find x from the equation $x^4 + 4x^3 - 8x = 32$.

If $4x^2 - 4x^2$ be inserted in the first side, we shall have—

$$x^4 + 4x^3 + 4x^2 - 4x^2 - 8x = 32.$$

That is, $(x^2 + 2x)^2 - 4(x^2 + 2x) = 32$.

Or, $y^2 - 4y = 32$, if $y = x^2 + 2x$.

Solving this, we find $y - 2 = \pm 6$, or $y = 2 \pm 6$.

$$\therefore \quad x^2 + 2x = 8 \text{ or } -4.$$

Two equations from which we obtain

$$x + 1 = \pm 3, \text{ or } \pm \sqrt{-3}.$$

And $\therefore\ x = 2$, or -4, or $\sqrt{-3} - 1$, or $-\sqrt{-3} - 1$.

As in No. 1, this equation contains the fourth power of x, and yields four values of it.

If it had contained the third power, and no higher, it would have given three values of x.

It will be found that, in every equation, the number of values of the unknown quantity corresponds to the highest power in which that quantity appears in the equation.

(4.) Solve the equation $x^2 - 5\sqrt{x^2 - 6x + 2} = 6x - 8$.

By transposing $6x - 2$, we have—

$$x^2 - 6x + 2 - 5\sqrt{x^2 - 6x + 2} = -6.$$

And if $y = \sqrt{x^2 - 6x + 2}$, the equation becomes $y^2 - 5y = -6$, from which we get $y = 3$ or 2; that is, $\sqrt{x^2 - 6x + 2} = 3$ or 2, and squaring, $x^2 - 6x + 2 = 9$ or 4: two quadratic equations, the first of which gives $x = 7$ or -1, and the second $x = 3 \pm \sqrt{11}$. So that x has again four values.

This appears to be contradictory to what was said after last example, for here the highest power of the unknown quantity is x^2. If, however, the equation be arranged so as to have the quantity under the root-sign on a side by itself, thus, $x^2 - 6x + 8 = 5\sqrt{x^2 - 6x + 2}$ (Art. 126), and if both sides be squared, we shall have—

$$x^4 - 12x^3 + 52x^2 - 96x + 64 = 25(x^2 - 6x + 2),$$

an equation of the fourth degree, and one therefore giving four values of x.

EXAMPLES FOR PRACTICE.—LV.

(1.) $x^4 - 29x^2 + 100 = 0$.

(2.) $4x^4 - 17x^2 + 18 = 0$.

(3.) $x^4 - 2x^3 + 2x^2 - x = 6$.

(4.) $x^4 + 2x^3 - 13x^2 = 14x - 24$.

(5.) $x - 2\sqrt{x} = 3$.

(6.) $5x + 7\sqrt{x} = 6$.

(7.) $3x + \sqrt{3x + 1} = 19$.

(8.) $x - \sqrt{4x - 3} = 6$.

(9.) $12x^2 + 2\sqrt{12x^2 + x - 2} = 10 - x$.

(10.) $x^2 + \frac{1}{x^2} - x + \frac{1}{x} = 2\frac{3}{4}$.

(11.) $x(x - a)(x - 2a)(x - 3a) = 1 - 2a^2$.

(12.) $(x^{2n} - 2x^n)^{2n} - 2(x^{2n} - 2x^n)^n + 1 = 0$.

155. Problems producing Quadratic Equations with One Unknown Quantity.

(1.) Find two numbers whose difference is 6 and whose product is 91.

Let x = the greater, and $x-6$ = the less,

Then $x(x-6)=91$, by the question.

Completing square, $x^2-6x+9=100$.

Extracting root, $x-3=\pm 10$.

$\therefore\ x=3\pm 10=13$ or -7.

And $x-6=7$ or -13.

The numbers are therefore 13 and 7 or -7 and -13, both of these pairs fulfilling the conditions of the question.

(2.) Bought at a sale a lot of sheep for £90, and found that their number exceeded the number of shillings each cost by 5. How many were there, and what was their price per head?

Let x = the number of sheep.

$x-5$ = the number of shillings each cost.

$x(x-5)$ = total cost in shillings.

$\therefore\ x^2-5x=1800$.

Solving, we find $x=45$ or -40,

And $x-5=40$ or -45.

The negative values being here inapplicable, the number bought was 45, and the price per head was 40s.

If, instead of "their number exceeded the number of shillings," we read "their number was less than the number of shillings," and again put $x=$ the number of sheep, we shall have the equation

$$x(x+5)=1800.$$

From which $x=40$ or -45.

And $x+5=45$ or -40.

The two questions are so related that the rejected value of x in the one equation becomes its true value in the other.

156. After solving a problem producing a quadratic equation, it will be necessary to ascertain which of the two values, always yielded by such equations, is applicable to the given problem. In some cases both will apply; and in those where only one does so, it will frequently be possible, by some alteration in the conditions of the question, to form a new problem, to which the other value will in its turn be applicable. "The reason of this seems to be that the algebraical mode of expression is more general than ordinary language" (*Wood's Algebra*).

(3.) A workman earned £5 in a certain time. If he had wrought six days more, and gained 10d. per day less, he would still have earned the same amount. How long did he work?

Let x = the number of days required; then $\frac{1200}{x}$ = what he earned per day in pence; $\frac{1200}{x+6}$ = what he would have earned per day on working six days more.

Therefore by the question,

$$\frac{1200}{x} = \frac{1200}{x+6} + 10.$$

Multiplying out and transposing,

$$x^2 + 6x - 720 = 0.$$

From which $x = 24$ or -30.

As -30 cannot be the answer, the number of days he wrought was 24.

The 30 which is here rejected, on account of its being negative, affords a solution of the following modification of the given problem:—

A workman earned £5 in a certain time. If he had wrought six days *less*, and gained 10d. per day *more*, he would still have earned the same amount. How many days did he work?

Putting x = the number of days required, we have the equation

$$\frac{1200}{x} + 10 = \frac{1200}{x-6}.$$

From which $x^2 - 6x - 720 = 0.$

And $\therefore\ x = 30 \text{ or } -24.$

(4.) Two persons entered into partnership. One put in £1000 for twelve months; the other continued in business for eight months more, and then retired with a sum of £1750. The total gain having been £350, what share of it should each have had?

Let x = the number of pounds the second gained; then $1750 - x$ = the number of pounds in his capital.

In a question of this kind it is understood of course that the profit is divided in proportion to the sums invested and the duration of their investment.

Now £1000 invested for twelve months may be considered equivalent to £12,000 invested for one month, and $1750 - x$ for 20 months the same as $20(1750 - x)$ for one month.

This gives $12000 + 20(1750 - x)$ as the total capital for one month, the share of the second partner being $20(1750 - x)$.

We have therefore by proportion

$$12000 + 20(1750 - x) : 20(1750 - x) :: 350 : x.$$

Cancelling 20 out of first and second terms, and reducing to an equation, we have—

$$(600 + 1750 - x)x = 350(1750 - x)$$

Or $x^2 - 2700x + 612500 = 0.$

This being solved, gives $x = 250$ or 2450.

As the entire gain was only £350, the larger value of x is here inapplicable.

The second partner's share was therefore £250, and the first's £100.

EXAMPLES FOR PRACTICE.—LVI.

(1.) Find two numbers whose sum is 18 and the sum of whose squares is 194.

(2.) A boat's crew take as many minutes to row one mile up a river as they can go miles down in one hour. If the river flows at the rate of $3\frac{1}{2}$ miles per hour, at what rate can they row in still water?

(3.) In a race between two steam-boats, one went one mile an hour faster than the other, and gained by six minutes. The distance being 24 miles, at what rate did each go?

(4.) Two numbers differ by one, and their cubes by 91; what are they?

(5.) The length of a rectangular picture-gallery is 30 feet greater than its width, and the walls are 20 feet high. Making no deductions for openings, the area of the walls is exactly equal to the joint area of the floor and the ceiling. What is the breadth of the room?

(6.) At a game of draughts, the number of pieces lost by one player was twice the number lost by the other; and it was observed that the product of these numbers exceeded the sum of the number of pieces left on the board by three. The loss of each piece having on an average occupied two minutes, it is required to find how long the game lasted. Each player began with twelve pieces.

(7.) A and B set out at the same time to meet one another from two places 87 miles apart. A travels 12 miles and B 11 miles a day more than the number of days which they require for the journey. How far does each travel?

(8.) A rectangular field of 3 acres is enclosed by a fence 484 yards long. Find the length and the breadth of the field.

(9.) In a certain town the consumption of water for domestic and trade purposes was 30 gallons a day for each inhabitant. By the adoption of measures to prevent waste, the domestic consumption was reduced two-sevenths and the trade consumption one-sixth. It was then found that the saving on the domestic consumption bore to that on the trade consumption the same proportion that the original consumption did to the total saving. To what was the total consumption reduced per head?

(10.) Three persons can together do a piece of work in 10 days. If each wrought by himself, A would take 10 days more than B, but only half as many as C. What time would each take?

(11.) The perimeter of a quadrangle is 280 feet. When a path 10 feet wide was formed round the inside of it, the area of the plot left in the centre was found to be half that of the whole quadrangle. Required its dimensions.

(12.) The number of passengers in a special train was the same as the number of miles in the journey. Thirty passengers paid one penny a mile, and of the remainder three-fourths paid twopence a mile, and one-fourth threepence a mile. The total amount paid being £15, 12s. 6d., what was the length of the journey?

(13.) For £6 a person can buy a certain number of yards of one kind of cloth, or 3 yards more of another kind worth 2s. a yard less. At what price per yard does each sell?

(14.) The expenses in a workman's club having increased to £60 per annum, it was found necessary either to increase the annual subscription by 1s. 6d., or to obtain 40 new members. What was the number of members and the annual subscription?

(15.) At an election, the number who voted for the unsuccessful candidate was a mean proportional between the number who voted for his opponent and those who did

not vote at all, of whom there were 432. He lost by 84 votes. What was the size of the constituency?

(16.) What time is it when the number of hours after noon is the same fraction of a day that an hour is of the number of hours it wants of 10 o'clock?

(17.) A farmer sold a number of geese and turkeys—18 in all—for £7, 14s. The number of geese was the same as the number of shillings a turkey cost, and the number of turkeys equalled the number of shillings a goose cost. How many were there of each?

(18.) A person went from X to Y by coach, and returned by train, the distance between the two places being 32 miles by road and 30 miles by rail. The rate of the train was 22 miles an hour greater than that of the coach, and it did the journey in 3 hours less. Find the time each took.

(19.) Of two bankrupts the first can pay per £1 what the other can pay per guinea. The debt of the first exceeding that of the second by £105, the creditors of each lose £210. Find the assets of the first and the debts of the second.

(20.) Two workmen received £4, 10s. for doing a piece of work between them in 4 days. If they had wrought separately, the one would have taken 6 days longer than the other. What share of the £4, 10s. should each have received?

(21.) One boatman can row 5 miles up a stream in an hour, while another can row only 4. On making a trip to a place situated 10 miles up, the first went and came in $36\frac{2}{3}$ minutes less than the second. Find the rate at which the stream flowed.

(22.) In balloting for the admission of members to a club, one black ball in 12 was sufficient to exclude. A new member was told that the number who voted against his admission, multiplied by itself, would only be equal to the number who voted for him; but that if 5 of these latter had voted against him, they would have increased

the dissentients to the number just sufficient to prevent his election. How many voted?

(23.) A bicyclist can make the same number of revolutions per minute with each of two machines, the first of which has the circumference of the driving-wheel 2 feet greater than the other. Using the first machine, he can go 3 miles per hour less than the number of feet in the circumference of its wheel, and using the second he can cover a mile in $4\frac{1}{2}$ minutes. Find the circumference of each of the wheels.

(24.) On the same day A and B withdrew all their money from the bank, A's having lain for 12 months and B's for 8. The bank-rate per cent. was exactly equal to one-half the difference between the sums they gained as interest, and the total of these sums was £41·28. The amounts withdrawn having been equal, it is required what each lodged in bank.

Selected Examples.

(25.) A person by selling a horse for £56 gains as much per cent. as the horse cost him. What did he pay for the horse?

(26.) Find the price of eggs per dozen when two less in a shilling's worth raises the price one penny per dozen.

(27.) Two detachments of foot, being ordered to a station at a distance of 39 miles, began their march at the same time; but the one party, by travelling a quarter of a mile an hour more than the other, arrived there an hour sooner. Find their rates of marching.

(28.) A person drew a quantity of wine from a full vessel which held 81 gallons, and then filled up the vessel with water. He then drew of the mixture as much as he before drew of pure wine, and it was found that 64 gallons of pure wine remained. How much did he draw each time?

(29.) A and B run a race. B, who runs slower than A by a mile in 5 hours, starts first by $2\frac{1}{2}$ minutes, and they get to the fifth mile-stone together. Required their rates of running.

(30.) A detachment from an army was marching in regular column, with 5 more men in depth than in front; but on the enemy coming in sight the front was increased by 845 men, and the whole was drawn up in 5 lines. Find the number of men in the detachment.

(31.) The product of four consecutive numbers is 3024. Find the numbers.

(32.) A cistern can be filled by two pipes running together in 2 hours 55 minutes. The larger pipe by itself will fill it sooner than the smaller by 2 hours. What time will each take separately to fill it?

(33.) A person lowered his goods a certain rate per cent., and found that to bring them back to the original price he must raise them $3\frac{1}{3}$ more per cent. than he had lowered them. Find the original fall per cent.

(34.) A grazier bought a certain number of oxen for £240, and, after losing 3, sold the remainder for £8 a head more than they cost him, and gained thereby £59. How many oxen did he buy?

(35.) A and B take shares in a concern to the amount altogether of £500. They sell out at par, A at the end of 2 years, B of 8, and each receives in capital and profit £297. How much did each embark?

(36.) Some bees were sitting on a tree. At one time there flew away a number of them represented by the square root of half the number in the swarm; and again, eight-ninths of the whole swarm; two only remaining. How many bees were there?—(*From the "Bija Ganita," a Hindu Treatise on Algebra.*)

157. Simultaneous Quadratic Equations.—Equations of two unknown quantities involving terms of two dimensions (Art. 145) have the following general form :—

$$ax^2 + bxy + cy^2 + dx + ey = f \qquad [1]$$
$$a_1x^2 + b_1xy + c_1y^2 + d_1x + e_1y = f_1 \qquad [2]$$

If [1] be multiplied by a_1, and [2] by a, we shall have—

$$a_1ax^2 + a_1bxy + a_1cy^2 + a_1dx + a_1ey = a_1f,$$
and $$a_1ax^2 + ab_1xy + ac_1y^2 + ad_1x + ae_1y = af_1.$$

From which, by subtraction and division, we get—

$$x = \frac{(ac_1 - a_1c)y^2 + (ae_1 - a_1e)y + (a_1f - af_1)}{(a_1b - ab_1)y + (a_1d - ad_1)}$$

If now this value of x be substituted in one of the original equations, we shall obtain an equation in y of the fourth degree: y therefore must have four values, and so consequently must x.

Although thus properly of the fourth degree, many equations of the above character, especially those in which some of the terms are wanting, may, by a little management, be solved as quadratics.

No general rule for their solution can be given, but the following examples will show some of the methods that may be employed.

158. I. When one of the equations is simple.

Find from the simple equation a value of one of the unknown quantities in terms of the other, and substitute this value in the other equation.

Illustrative Example.

Given $\left\{\begin{array}{rl} x^2 - 2xy - 3y^2 = 12 & [1] \\ x - 2y = 3 & [2] \end{array}\right\}$, to find x and y.

From [2], $x = 2y + 3.$

Substitute in [1]—

$$(2y+3)^2 - 2(2y+3)y - 3y^2 = 12.$$

Then $$-3y^2 + 6y + 9 = 12.$$

$$y^2 - 2y + 1 = 0.$$

And $$y - 1 = 0.$$

$$\therefore\ y = 1, \text{ and } x = 2y + 3 = 5.$$

Observe here that only one value of each of the unknown quantities has been found, but if these be successively substituted in the first equation, we shall obtain, when $y = 1$, two values for x,—namely, 5 and -3; and when $x = 5$, two values for y,—namely, 1 and $-4\frac{1}{3}$.

These second values, -3 and $-4\frac{1}{3}$, however, do not satisfy the second equation, so that only the first found pair of values is applicable to both equations.

159. II. When the equations are symmetrical; that is, when the unknown quantities in each are similarly involved.

Assume $x = u + v$, and $y = u - v$; substitute these values in the given equations, and therefrom deduce a quadratic equation in u. This being solved, v and therefore x and y may be found.

Illustrative Example.

Given $\left\{\begin{array}{ll} 2x^2 - 3xy + 2y^2 = 23 & [1] \\ x^2 - x + y^2 - y = 26 & [2] \end{array}\right\}$, to find x and y.

Let $x = u + v$, $y = u - v$.

Then $2(u+v)^2 - 3(u+v)(u-v) + 2(u-v)^2 = 23$ [3]

And $(u+v)^2 - (u+v) + (u-v)^2 - (u-v) = 26$ [4]

From [3], $u^2 + 7v^2 = 23$ [5]

From [4], $u^2 - u + v^2 = 13$ [6]

[6] × 7, $7u^2 - 7u + 7v^2 = 91$ [7]

[7] − [5], $6u^2 - 7u = 68$

$$\therefore\ u = 4 \text{ or } -\frac{17}{6}.$$

And from [5] or [6], $v = \pm 1$ or $\pm\frac{\sqrt{77}}{6}$.

$$\therefore\ x = 5 \text{ or } 3 \text{ or } \frac{\pm\sqrt{77} - 17}{6}.$$

And $$y = 3 \text{ or } 5 \text{ or } \frac{\mp\sqrt{77} - 17}{6}.$$

160. III. When the equations are homogeneous; that is, when all the terms in each, except the known one, are of the same dimension.

Assume $y = vx$, and substitute this value of y in the given equations. From each, find x^2 in terms of v, and equate the two expressions thus obtained.

This gives a quadratic in v; and v being found, the original equations may easily be solved.

Illustrative Example.

Given $\left\{\begin{matrix} x^2 + xy - 2y^2 = 7 & [1] \\ x^2 + 3y^2 = 21 & [2] \end{matrix}\right\}$, to find x and y.

Let $$y = vx$$

From [1], $$x^2 = \frac{7}{1 + v - 2v^2}$$

From [2], $$x^2 = \frac{21}{1 + 3v^2}$$

$$\therefore\ \frac{7}{1 + v - 2v^2} = \frac{21}{1 + 3v^2}$$

And $$1 + 3v^2 = 3 + 3v - 6v^2$$

Or $$9v^2 - 3v = 2$$

From which $$v = \frac{2}{3} \text{ or } -\frac{1}{3}$$

$$\therefore\ y = \frac{2}{3}x \text{ or } -\frac{1}{3}x.$$

Substituting the first of these values in [1] we get—

$$x^2 + \frac{2}{3}x^2 - \frac{8}{9}x^2 = 7.$$

From which, $x = \pm 3$, and $\therefore\ y = \pm 2$.

Substituting the second value of y in [1] we get—

$$x^2 - \frac{1}{3}x^2 - \frac{2}{9}x^2 = 7.$$

From which, $x = \pm\frac{3}{2}\sqrt{7}$, and $\therefore\ y = \mp\frac{1}{2}\sqrt{7}$.

Making the same substitutions in [2], we obtain the same values of x and y, which have, therefore, each four values,—namely,

$$x = \pm 3 \text{ or } \pm\frac{3}{2}\sqrt{7}$$

$$\text{and } y = \pm 2 \text{ or } \mp\frac{1}{2}\sqrt{7}.$$

161. By employing the following modification of the above method, x may be expressed in terms of y, or y in terms of x, without employing the auxiliary quantity v:—

$$x^2 + xy - 2y^2 = 7 \quad [1]$$
$$x^2 + 3y^2 = 21 \quad [2]$$

Find the least common multiple of the two *known* quantities, and multiply each equation by such a number as will bring its *known* quantity to the common multiple found.

The above will then become—

$$3x^2 + 3xy - 6y^2 = 21$$
$$x^2 + 3y^2 = 21$$

Subtract the one equation from the other; this gives—

$$2x^2 + 3xy - 9y^2 = 0 \quad [3]$$

Resolving into factors, we have $(2x-3y)(x+3y)=0$.

$$\therefore\ y=\frac{2}{3}x \text{ or } -\frac{1}{3}x.$$

Substituting these values in [1] or [2] we get, as before—

$$x=\pm 3 \text{ or } \pm\frac{3}{2}\sqrt{7}$$

$$\text{and } y=\pm 2 \text{ or } \mp\frac{1}{2}\sqrt{7}.$$

If the quadratic equation [3] is solved by completing the square, one of the unknown quantities must be considered the coefficient of the other.

Thus, solving for y in terms of x, we have—

From [3], $9y^2-3xy=2x^2$

$$\text{or } y^2-\frac{1}{3}xy=\frac{2}{9}x^2$$

And completing square—

$$y^2-\frac{1}{3}xy+\left(\frac{1}{6}x\right)^2=\frac{x^2}{36}+\frac{2x^2}{9}=\frac{9x^2}{36}.$$

$$\therefore\ y=\frac{1\pm 3}{6}x=\frac{2}{3}x \text{ or } -\frac{1}{3}x, \text{ as before.}$$

Additional Example.

Given $\left\{\begin{matrix}(2x-3y)x=5 & [1]\\ (3x-4y)y=9 & [2]\end{matrix}\right\}$, to find x and y.

[1] × 9, $18x^2-27xy=45$

[2] × 5, $15xy-20y^2=45$

Subtracting, $18x^2-42xy+20y^2=0$ [3]

Solving for x in terms of y—

[3] ÷ 18, $x^2-\frac{7}{3}xy=-\frac{10y^2}{9}$

Completing square, $x^2-\frac{7}{3}xy+\frac{49}{36}y^2=\frac{49-40}{36}y^2=\frac{9}{36}y^2$

$$\therefore\ x=\frac{7\pm 3}{6}y=\frac{5}{3}y \text{ or } \frac{2}{3}y.$$

Substituting these values successively in [1]—

First, $\frac{50}{9}y^2 - \frac{15}{3}y^2 = 5, \quad \therefore \ y = \pm 3, \text{ and } x = \pm 5.$

Second, $\frac{8}{9}y^2 - \frac{6}{3}y^2 = 5, \quad \therefore \ y = \pm 3\sqrt{-\frac{1}{2}}$

and $x = \pm 2\sqrt{-\frac{1}{2}}.$

162. IV. When the given equations will admit, they should be solved so as to find expressions for the sum and difference of the unknown quantities.

Illustrative Examples.

(1.) Given $\left\{\begin{matrix} x^2+y^2=109 & [1] \\ x+y=13 & [2] \end{matrix}\right\}$, to find x and y.

Squaring [2], $x^2+2xy+y^2=169$ [3]
[3] − [1], $2xy=60$ [4]
[1] − [4], $x^2-2xy+y^2=49$ [5]
Extracting root, $x-y=\pm 7$ [6]
[2] + [6], $2x=20$ or 6, and $\therefore \ x=10$ or 3.
[2] − [6], $2y=6$ or 20, and $\therefore \ y=3$ or 10.

(2.) Given $\left\{\begin{matrix} x^2+y^2=89 & [1] \\ xy+x-y=43 & [2] \end{matrix}\right\}$, to find x and y.

[2] × 2, $2xy+2(x-y)=86$ [3]
[1] − [3], $x^2-2xy+y^2-2(x-y)=3$
or $(x-y)^2-2(x-y)=3$ [4]
Com. sq., $(x-y)^2-2(x-y)+1=4$
Extracting root, $x-y-1=\pm 2$
$\therefore \ x-y=3 \text{ or } -1$ [5]
[2] − [5], $xy=40$ or 44
and $2xy=80$ or 88 [6]
[1] + [6], $x^2+2xy+y^2=169$ or 177
Extracting root, $x+y=\pm 13$ or $\pm\sqrt{177}$ [7]

[7] + [5], $2x = 16 \text{ or } -10 \text{ or } -1 \pm \sqrt{177}$

$$\therefore\ x = 8 \text{ or } -5 \text{ or } \frac{-1 \pm \sqrt{177}}{2}$$

[7] − [5], $2y = 10 \text{ or } -16 \text{ or } 1 \pm \sqrt{177}$

$$\therefore\ y = 5 \text{ or } -8 \text{ or } \frac{1 \pm \sqrt{177}}{2}.$$

(3.) Given $\left\{\begin{matrix} x^2+y^2=x^2y^2 & [1] \\ m(x+y)=xy & [2] \end{matrix}\right\}$, to find x and y.

[2] squared,

$$m^2(x^2+2xy+y^2)=x^2y^2 \qquad [3]$$

[1] × m^2, $\quad m^2(x^2+y^2)=m^2x^2y^2 \qquad [4]$

[3] − [4], $\quad 2m^2xy=(1-m^2)x^2y^2$

$$\therefore\ xy=\frac{2m^2}{1-m^2} \text{ or } 0 \qquad [5]$$

Substitut. in [1], $\quad x^2+y^2=\frac{4m^4}{(1-m^2)^2} \text{ or } 0 \qquad [6]$

[5] × 2, $\quad 2xy=\frac{4m^2}{1-m^2} \qquad [7]$

[6] + [7], $\quad x^2+2xy+y^2=\frac{4m^4+4m^2(1-m^2)}{(1-m^2)^2}=\frac{4m^2}{(1-m^2)^2}$

$$\text{or } x+y=\frac{\pm 2m}{1-m^2}$$

[6] − [7], $\quad x^2-2xy+y^2=\frac{4m^4-4m^2(1-m^2)}{(1-m^2)^2}=\frac{4m^2(2m^2-1)}{(1-m^2)^2}$

$$\text{or } x-y=\frac{\pm 2m\sqrt{2m^2-1}}{1-m^2}$$

$$\therefore\ x=\pm m(1+\sqrt{2m^2-1})$$

$$\text{and } y=\pm m(1-\sqrt{2m^2-1}).$$

If $xy=0$ be substituted in [1] it will be found that x and y each equal 0.

163. Equations containing terms of more than two dimensions.

Illustrative Examples.

(1.) Given $\left\{\begin{array}{ll} x^3-y^3=63 & [1] \\ x^2+xy+y^2=21 & [2] \end{array}\right\}$, to find x and y.

$[1] \div [2]$, $\quad x-y=3 \qquad [3]$

$[3]$ squared, $\quad x^2-2xy+y^2=9 \qquad [4]$

$[2]-[4]$, $\quad 3xy=12$ and $xy=4 \qquad [5]$

$[2]+[5]$, $\quad x^2+2xy+y^2=25$

or $\quad x+y=\pm 5 \qquad [6]$

$[6]+[3]$, $\quad \therefore\ 2x=8$ or -2, $\quad \therefore\ x=4$ or -1.

$[6]-[3]$, $\quad 2y=2$ or -8, $\quad \therefore\ y=1$ or -4.

(2.) Given $\left\{\begin{array}{ll} x^4+y^4=97 & [1] \\ x+y=5 & [2] \end{array}\right\}$, to find x and y.

From $[2]$, $\quad x^4+4x^3y+6x^2y^2+4xy^3+y^4=625 \qquad [3]$

$[3]-[1]$, $\quad 4x^3y+6x^2y^2+4xy^3=528$

or $\quad 2xy(x^2+y^2)+3x^2y^2=264 \qquad [4]$

From $[2]$, $\quad x^2+y^2=25-2xy$

Substitut. in $[4]$, $\quad 2xy(25-2xy)+3x^2y^2=264$

$$50xy-4x^2y^2+3x^2y^2=264$$

$$x^2y^2-50xy+264=0$$

$$(xy-6)(xy-44)=0$$

$$\therefore\ xy=6 \text{ or } 44.$$

$x+y$ being known, $x-y$ may now be found, and therefore x and y; their values are—

$$x=3 \text{ or } 2 \text{ or } \tfrac{1}{2}(5\pm\sqrt{-151})$$

$$y=2 \text{ or } 3 \text{ or } \tfrac{1}{2}(5\mp\sqrt{-151}).$$

(3.) Given $\left\{\begin{array}{ll} x^3+2x^2y+4xy^2+8y^3=32 & [1] \\ x^4y^2+4x^2y^4=32 & [2] \end{array}\right\}$ to find x and y.

n [1], $(x^2+4y^2)(x+2y)=32$ [3]

n [2], $(x^2+4y^2)x^2y^2=32$ [4]

÷ [4], $\frac{x+2y}{x^2y^2}=1$, or $x+2y=x^2y^2$ [5]

quared, $x^2+4xy+4y^2=x^4y^4$

or $x^2+4y^2=x^4y^4-4xy$ [6]

titut. in [4], $(x^4y^4-4xy)x^2y^2=32$

$x^6y^6-4x^3y^3=32$ [7]

pl. square, $x^6y^6-4x^3y^3+4=36$

$\therefore\ x^3y^3=2\pm6=8$ or -4

and $xy=2$ or $-\sqrt[3]{4}$ [8]

tituting the first of these values in [4], we get—

$(x^2+4y^2)(4)=32$, or $x^2+4y^2=8$ [9]

n [8] and [9] $\begin{cases} x^2+4xy+4y^2=16 \\ x^2-4xy+4y^2=0 \end{cases}$

$\therefore\ x+2y=\pm4$

and $x-2y=0$

$\therefore\ x=\pm2$, and $y=\pm1$.

.her values of x and y may also be found; for, resolving nto factors, we get—

$\cdot 2)(xy+\sqrt[3]{4})(x^2y^2+2xy+4)(x^2y^2-xy\sqrt[3]{4}+2\sqrt[3]{2})=0.$

on equating these successively to 0, we obtain—

$=2$, or $-\sqrt[3]{4}$, or $\pm\sqrt{-3}-1$, or $\frac{1}{2}\{\sqrt[3]{4}\pm\sqrt[6]{-432}\}$,

x different values of xy which, in conjunction with yield twelve values of x and twelve of y.

EXAMPLES FOR PRACTICE.—LVII.

nd the values of x and y in the following equations :—

(1.) $x-y=3$, $x^2-y^2=33$.

(2.) $x+y=7$, $x^2+y^2=29$.

(3.) $x^2+y=11$, $3x-y=7$.

(4.) $2x^2+y^2=9$, $2x+y=5$.

(5.) $x^2+y=17$, $x^2y=16$.

(6.) $x^2+9y^2=61,\ xy=10.$

(7.) $x^2+xy+y^2=7,\ x+y=3.$

(8.) $x^2+xy=70,\ xy+y^2=30.$

(9.) $\cdot 01x^2-\cdot 2y=1,\ \cdot 3x-\dfrac{4y}{x}=3.$

(10.) $3x^2-5xy=8,\ 2xy+3y^2=28.$

(11.) $\dfrac{1}{x}+\dfrac{1}{y}=\dfrac{5}{12},\ xy=24.$

(12.) $\dfrac{x}{y}+\dfrac{y}{x}=\dfrac{25}{12},\ x-y=2.$

(13.) $\dfrac{1}{x^2}+\dfrac{4}{y^2}=25,\ \dfrac{1}{x}+\dfrac{2}{y}=7.$

(14.) $x^3+y^3=28,\ x+y=4.$

(15.) $x^3-y^3=98,\ x^2y-xy^2=30.$

(16.) $x^3+y^3=407,\ x^2-xy+y^2=37.$

(17.) $x^4+x^2y^2+y^4=17\frac{1}{16},\ x^2+xy+y^2=5\frac{1}{4}.$

(18.) $16x^4y^2+16x^2y^4=333,\ 4x^2y+4xy^2=21.$

(19.) $\dfrac{x^4}{y^2}-5\dfrac{x^2}{y}=234,\ x^2+y^2=40.$

(20.) $\dfrac{x^2}{y^2}+\dfrac{y^2}{x^2}-\dfrac{x}{y}+\dfrac{y}{x}=2\frac{3}{4},\ x-y=1.$

(21.) $x^2+y^2=85,\ xy+x+y=29.$

(22.) $x^4+y^4=257,\ x+y=5.$

(23.) $\dfrac{x+y}{x-y}+\dfrac{x-y}{x+y}=10,\ x^2y^2=6.$

(24.) $x^2y^2=16xy+420,\ x^2+y^2=61.$

(25.) $\dfrac{x}{y}+\dfrac{y}{x}=2,\ x+y=a.$

(26.) $x^2-3xy=m,\ 3x+y=n.$

(27.) $\dfrac{a}{x}+\dfrac{a+b}{y}=a+2b,\ xy=\dfrac{a}{b}.$

(28.) $x^2+y^2+x+y=2(a^2+a+1)$, $xy=a^2-1$.

(29.) $xy+xy^2=ab+b^2$, $x+xy^3=a^2+\frac{b^3}{a}$.

(30.) $x^2y^2-2ab^3xy=a^2b^2(a^4-b^4)$, $x^2-4y^2=(a^2-b^2)^2$.

It has not been considered necessary to give all the values of x and y that may be found from some of the above equations.

164. Three Unknown Quantities. — Occasionally the methods employed in the previous article may be used to solve equations with three unknown quantities of the second or higher dimensions.

Illustrative Examples.

(1.) Find values of x, y, and z that will satisfy the following equations:—

$$(x+y)(y+z)=a^2 \qquad [1]$$
$$(y+z)(z+x)=b^2 \qquad [2]$$
$$(z+x)(x+y)=c^2 \qquad [3]$$

Multiply the equations together—

$$(x+y)^2(y+z)^2(z+x)^2=a^2b^2c^2 \qquad [4]$$
$$\therefore\ (x+y)(y+z)(z+x)=\pm abc \qquad [5]$$

$[5]\div[1]$, $\qquad z+x=\pm\frac{bc}{a}$ $\qquad [6]$

$[5]\div[2]$, $\qquad x+y=\pm\frac{ac}{b}$ $\qquad [7]$

$[5]\div[3]$, $\qquad y+z=\pm\frac{ab}{c}$ $\qquad [8]$

$$[6]+[7]+[8],\ 2(x+y+z)=\pm\left(\frac{bc}{a}+\frac{ac}{b}+\frac{ab}{c}\right)$$
$$\therefore\ x+y+z=\pm\frac{1}{2}\left(\frac{bc}{a}+\frac{ac}{b}+\frac{ab}{c}\right) \qquad [9]$$

$$[9] - [8], \qquad x = \pm\frac{1}{2}\left(\frac{bc}{a} + \frac{ac}{b} - \frac{ab}{c}\right),$$

$$[9] - [7], \qquad y = \pm\frac{1}{2}\left(\frac{bc}{a} + \frac{ab}{c} - \frac{ac}{b}\right),$$

$$[9] - [6], \qquad z = \pm\frac{1}{2}\left(\frac{ab}{c} + \frac{ac}{b} - \frac{bc}{a}\right).$$

(2.) Given $\left\{\begin{array}{rl} x^2+y^2+z^2 = 38 & [1] \\ (x+y+z)y = 30 & [2] \\ xz = 10 & [3] \end{array}\right\}$ to find x, y, and z.

From [2], $2xy + 2yz = 60 - 2y^2$ [4]

From [3], $2xz = 20$ [5]

[1] + [4] + [5], $x^2+y^2+z^2+2xy+2yz+2xz = 118 - 2y^2$

or $(x+y+z)^2 = 118 - 2y^2$ [6]

[6] × y^2, $(x+y+z)^2y^2 = (118 - 2y^2)y^2$

[2] squared, $(x+y+z)^2y^2 = 900$

$\therefore (118 - 2y^2)y^2 = 900$

From which, $y^2 = 9$ or 50

Using the first of these values, we obtain—

From [1], $x^2 + z^2 = 29$

From [3], $2xz = 20$

$\therefore (x+z)^2 = 49$, or $x+z = \pm 7$

and $(x-z)^2 = 9$, or $x \sim z = \pm 3$

We have $\therefore x = \pm 5$, $y = \pm 3$, and $z = \pm 2$.

From $y^2 = 50$ we may obtain other values of x and z, but they will be found to be imaginary.

(3.) Find x, y, and z from the equations—

$$\begin{array}{rl} x^2+y^2+z^2 = a & [1] \\ xy+xz+yz = b & [2] \\ xy(x+y+z) = c & [3] \end{array}$$

From [1] and [2] we have

$$x^2+y^2+z^2+2(xy+xz+yz) = a+2b$$

Extracting root, $x+y+z = \pm\sqrt{a+2b}$ [4]

$[3] \div [4]$, $\qquad xy = \pm \dfrac{c}{\sqrt{a+2b}}$ $\qquad [5]$

From [2] and [5] we have—

$$2xy - 2xz - 2yz = \pm \frac{4c}{\sqrt{a+2b}} - 2b \qquad [6]$$

$$[1]+[6], \quad x^2+y^2+z^2+2(xy-xz-yz) = a \pm \frac{4c}{\sqrt{a+2b}} - 2b$$

$$\text{Extracting root, } x+y-z = \pm\sqrt{a \pm \frac{4c}{\sqrt{a+2b}} - 2b} \qquad [7]$$

From [4] and [7] z may be found, and consequently, in conjunction with [5], we may obtain values of x and y.

EXAMPLES FOR PRACTICE.—LVIII.

Find the values of x, y, and z from the following equations :—

(1.) $xy = 20$, $xz = 24$, $yz = 30$.

(2.) $x^2yz = ab$, $xy^2z = bc$, $xyz^2 = ac$.

(3.) $(x+y)(y+z) = 21$, $(y+z)(z+x) = 42$, $(z+x)(x+y) = 18$.

(4.) $x(x+y+z) = a$, $y(x+y+z) = b$, $z(x+y+z) = c$.

(5.) $(x+y)z = 60$, $(x+z)y = 78$, $(y+z)x = 90$.

(6.) $10xyz = 20(x+y) = 15(x+z) = 12(y+z)$.

(7.) $\dfrac{x}{a} + \dfrac{b}{y} = 3$, $\dfrac{y}{b} + \dfrac{c}{z} = 3\frac{1}{2}$, $\dfrac{z}{c} + \dfrac{a}{x} = 1\frac{1}{3}$.

(8.) $xy + 3xz + 5yz = 35$, $2xy - xz + 3yz = 21$, $3xy - 2xz - yz = 18$.

(9.) $x^2 + xy + y^2 = 37$, $y^2 + yz + z^2 = 13$, $z^2 + xz + x^2 = 21$.

(10.) $x+y+z = 10$, $x^2+y^2+z^2 = 38$, $xy+yz = 21$.

(11.) $x^2+y^2+z^2 = a^2 - 2b^2$, $xy+xz+yz = b^2$, $xz+yz+z^2 = ab$.

(12.) $x^2+y^2+z^2 = 109$, $x^2 - yz = 46$, $y^2 - xz = 12$.

$[3] \div [4]$, $\qquad xy = \pm \dfrac{c}{\sqrt{a+2b}}$ $\qquad [5]$

From [2] and [5] we have—

$$2xy - 2xz - 2yz = \pm \frac{4c}{\sqrt{a+2b}} - 2b \qquad [6]$$

$[1]+[6]$, $\quad x^2+y^2+z^2+2(xy-xz-yz) = a \pm \dfrac{4c}{\sqrt{a+2b}} - 2b$

Extracting root, $x+y-z = \pm\sqrt{a \pm \dfrac{4c}{\sqrt{a+2b}} - 2b}$ $\quad [7]$

From [4] and [7] z may be found, and consequently, in conjunction with [5], we may obtain values of x and y.

EXAMPLES FOR PRACTICE.—LVIII.

Find the values of x, y, and z from the following equations:—

(1.) $xy = 20$, $xz = 24$, $yz = 30$.

(2.) $x^2yz = ab$, $xy^2z = bc$, $xyz^2 = ac$.

(3.) $(x+y)(y+z) = 21$, $(y+z)(z+x) = 42$, $(z+x)(x+y) = 18$.

(4.) $x(x+y+z) = a$, $y(x+y+z) = b$, $z(x+y+z) = c$.

(5.) $(x+y)z = 60$, $(x+z)y = 78$, $(y+z)x = 90$.

(6.) $10xyz = 20(x+y) = 15(x+z) = 12(y+z)$.

(7.) $\dfrac{x}{a} + \dfrac{b}{y} = 3$, $\dfrac{y}{b} + \dfrac{c}{z} = 3\frac{1}{2}$, $\dfrac{z}{c} + \dfrac{a}{x} = 1\frac{1}{3}$.

(8.) $xy + 3xz + 5yz = 35$, $2xy - xz + 3yz = 21$, $3xy - 2xz - yz = 18$.

(9.) $x^2 + xy + y^2 = 37$, $y^2 + yz + z^2 = 13$, $z^2 + xz + x^2 = 21$.

(10.) $x+y+z = 10$, $x^2+y^2+z^2 = 38$, $xy+yz = 21$.

(11.) $x^2+y^2+z^2 = a^2 - 2b^2$, $xy+xz+yz = b^2$, $xz+yz+z^2 = ab$.

(12.) $x^2+y^2+z^2 = 109$, $x^2-yz = 46$, $y^2-xz = 12$.

165. Problems producing Quadratic Equations of Two or more Unknown Quantities.

Illustrative Examples.

(1.) Two farmers took to market between them 600 bushels of oats. Each received the same sum for his share of them; but if the first had sold his at the rate the second did, and the second his at the rate the first did, then the one would have received £40, 0s. 10d., and the other £35, 0s. 10d. How many bushels did each sell, and what price per bushel did each receive?

Let x = number of bushels the first sold.

Then $600 - x$ = number of bushels the second sold.

Also let y = sum each received in pence.

Then $\frac{y}{x}$ = price a-bushel at which the first sold.

And $\frac{y}{600-x}$ = price a-bushel at which the second sold.

Then by the question, we have—

$$\frac{xy}{600-x} = 9610 \text{ [1], and } \frac{(600-x)y}{x} = 8410 \text{ [2].}$$

[1] × [2], $y^2 = 9610 \times 8410$

$$\therefore\ y = 31 \times 29 \times 10 = 8990\text{d.} = £37,\ 9\text{s. } 2\text{d.}$$

[1] ÷ [2], $\frac{x^2}{(600-x)^2} = \frac{961}{841}$

$$\therefore\ \frac{x}{600-x} = \frac{31}{29} \text{ or } \frac{x}{600} = \frac{31}{60}$$

$$\therefore\ x = 310, \text{ and } 600 - x = 290.$$

Also $\frac{y}{x} = \frac{8990}{310} = 29\text{d.}$

And $\frac{y}{600-x} = \frac{8990}{290} = 31\text{d.}$

So that the one had 310 bushels at 2s. 5d. per bushel, and the other 290 bushels at 2s. 7d. per bushel.

(2.) A and B set out from the same place, at the same time, to ascend a mountain. A travels at a uniform pace, and reaches the summit in 4 hours. B, after travelling one-half mile more than half the distance, reduces his pace by three-fourths of a mile per hour, and arrives at the top 12 minutes after A. In returning, each maintains a pace half a mile per hour greater than that at which he began the ascent, and it is found that B descends in $\frac{14}{15}$ths of the time A takes. Find the rate at which each began the ascent.

Let x = number of miles A takes per hour in ascending.

Then $4x$ = distance to top of mountain.

Let y = number of miles B goes at first.

Then $\dfrac{2x+\frac{1}{2}}{y}$ = number of hours B takes to first part.

And $\dfrac{2x-\frac{1}{2}}{y-\frac{3}{4}}$ = number of hours B takes to second part.

Then by the question, $$\frac{2x+\frac{1}{2}}{y}+\frac{2x-\frac{1}{2}}{y-\frac{3}{4}}=4\tfrac{1}{5}. \qquad [1]$$

Also on returning, $$\frac{4x}{y+\frac{1}{2}}=\frac{14}{15}\left(\frac{4x}{x+\frac{1}{2}}\right). \qquad [2]$$

From [2], $y=\dfrac{1}{28}(30x+1)$.

By substituting this value of y in [1], and reducing, we obtain the equation—

$$6x^2-19x+3=0.$$

From which we have, after solution—

$$x=3 \text{ or } \frac{1}{6}$$

$$\text{and } \therefore\ y=3\tfrac{1}{4} \text{ or } \frac{3}{14}.$$

As $2x$ must be greater than $\frac{1}{2}$, and y greater than $\frac{3}{4}$, the second values are inapplicable.

(3.) There are three numbers such that the product of the first by the sum of the other two is 408, of the second by the sum of the other two is 364, and of the third by the sum of the first two is 330. Find them.

Let x, y, and z be the numbers in order.
Then by the question,

$$x(y+z)=408 \quad [1]$$
$$y(x+z)=364 \quad [2]$$
$$z(x+y)=330 \quad [3]$$

By addition, $2(xy+xz+yz)=1102$

or $xy+xz+yz=551 \quad [4]$

[4] − [3], $xy=221 \quad [5]$

[4] − [2], $xz=187 \quad [6]$

[4] − [1], $yz=143 \quad [7]$

By multiplication, $x^2y^2z^2=221\times187\times143$

$$=17\times13\times17\times11\times13\times11$$

Extracting root, $xyz=\pm17\times13\times11 \quad [8]$

[8] ÷ [7], $x=\pm\dfrac{17\times13\times11}{13\times11}=\pm17$

[8] ÷ [6], $y=\pm\dfrac{17\times13\times11}{17\times11}=\pm13$

[8] ÷ [5], $z=\pm\dfrac{17\times13\times11}{17\times13}=\pm11$

EXAMPLES FOR PRACTICE.—LIX.

(1.) A ferryman took three more passengers than he was licensed to carry, and thereby drew as much money as if he had carried only the full complement, but had charged each a penny more than the usual fare. For this he was fined one guinea, by which he lost twice what he drew for the trip. How many was he licensed to carry, and what was the fare?

(2.) A grocer sold 5 lbs. of tea and 6 lbs. of coffee for 25 shillings, and he gave 2 lbs. more of coffee for 10 shillings than of tea for 12 shillings. What was the price of each per lb.?

(3.) The fore wheel of a carriage makes 12 revolutions more than the hind wheel in going a distance of 117 yards; but if each be increased one foot in circumference, the fore wheel will only make 12 more in going 140 yards. Find the circumference of each.

(4.) Two lads running a race start from opposite ends of the course at the same time. When they meet, it is found that one has run 220 yards further than the other, and that they will complete the remaining parts of their course in 4 minutes and $6\frac{1}{4}$ minutes respectively. Required their rates of running, and the length of the course.

(5.) A cistern was filled in 6 hours by two pipes running into it together; but if the pipes had been such that each when running alone would have required 5 hours less to fill the cistern than it did, then when running together they would have taken only 3 hours and 20 minutes. What time would each have taken separately?

(6.) A pedestrian set out to go from A to B, and when he had gone $16\frac{1}{2}$ miles, he was passed by a coach which had left A 4 hours after him; $2\frac{1}{2}$ hours later, 10 miles from B, he met the same coach returning from B. Neither of them having made any stoppages, it is required to ascertain their rates of travelling, and the distance between A and B.

(7.) The difference between a certain fraction and its reciprocal is $\frac{7}{12}$, while that between their squares is $\frac{175}{144}$. Find the fraction.

(8.) A draper bought three pieces of cloth for £125, paying for each as many shillings per yard as there were

yards in its length. Had each piece been a yard longer, and cost a shilling more per yard, he would have paid £8, 15s. more for the whole; but if the first had cost per yard what the second did, and the second what the first did, while the price of the third remained unaltered, he would have saved 4 shillings. How many yards were in each piece?

(9.) A, B, and C undertake a piece of work which they can together do in 4 days. At the close of the first day C leaves, but A and B go on for 3 days more, when B also leaves, and it takes A other 2 days to finish. Had A been able to lessen by 1 day the time in which he could have himself done the whole work, and had B similarly lessened his time for the whole by 3 days, the two could have finished it between them in $4\frac{1}{2}$ days. How long would each take separately to the whole?

(10.) The area of a square exceeds that of a rectangle by 18 square feet. If the dimensions of both figures be increased by 6 feet, the difference between the areas will be increased by 6 square feet; if all the dimensions be diminished by 6 feet, the rectangle will be two-thirds of the size of the square. Find the dimensions of each.

(11.) A, B, and C set out simultaneously from the same point to walk round an island 30 miles in circumference. A, proceeding in a direction contrary to that of the other two, met B and then C at an interval of one hour. C then, doubling his pace, overtook B in other $7\frac{1}{2}$ hours; whereas, if he had continued at the same rate as before, B would have overtaken him 20 hours after the first start. At what rate did each travel?

(12.) A sum of money laid out at simple interest for as many years as there are pounds in the rate per cent. amounts to a pounds; and the present value of the same

sum for the same rate and time is b pounds. What is the sum, and for how long is it lent?

Selected Examples.

(13.) Two cubical vessels together hold 407 cubic inches; when one is placed on the other, the total height is 11 inches. Find the contents of each.

(14.) A person buys two bales of cloth, each containing 80 yards, for £60. By selling the first at a *gain* and the second at a *loss* of as much per cent. as the second bale cost him, he finds he has made a profit of £4 on the whole. Required the cost price per yard of each bale.

(15.) A and B run a race round a two-mile course. In the first heat, B reaches the winning post two minutes before A. In the second heat, A increases his speed 2 miles per hour, and B diminishes his as much, and then A arrives at the winning post two minutes before B. Find at what rate each man ran in the first heat.

(16.) Find two numbers such that their sum multiplied by the sum of their squares shall be 272, while their difference multiplied by the difference of their squares shall be 32.

(17.) A person has £13,000, which he divides into two parts, and placing them at interest, receives an equal income from each. If he placed the first sum at the rate of interest of the second, he would receive £360 income; and if he placed the second sum at the rate of the first, he would receive £490 income. What are the two sums, and what the rates of interest?

(18.) A and B are two towns situated 18 miles apart on the same bank of a river. A man goes from A to B in 4 hours, by rowing the first half of the distance and walking the second half. In returning, he walks the first half at the same rate as before, but the stream being with

him, he rows $1\frac{1}{2}$ mile per hour more than in going, and accomplishes the whole distance in $3\frac{1}{2}$ hours. Find his rates of walking and rowing.

(19.) A person bought a number of sheep for £13, 10s. Reserving a portion of them, he sold the remainder for the same sum that the whole had cost him, and had a profit per head on those sold of as many shillings as the number of the sheep he had reserved. Now he found that had he reserved one sheep less, his profit per head would have been 1s. $10\frac{1}{2}$d. Required the number of sheep bought, and the number reserved.

(20.) What two numbers are those whose sum multiplied by the greater is 209, and whose difference multiplied by the less is 24?

(21.) A sets out to walk to a town $10\frac{1}{2}$ miles distant; and 30 minutes afterwards B is sent after him, overtakes him, and then returns to the place they started from at the same time that A reaches the town. If B walk 4 miles an hour, what is A's rate?

(22.) If £300 be laid out at simple interest for a certain number of years, it will amount to £360. If the same be allowed to remain two years longer, and at a rate of interest one per cent. higher, it will amount to £405. Required the rate per cent., and the number of years on the first supposition.

(23.) There are two numbers such that 3 times the sum of their squares multiplied by the less is equal to 26 times the greater, and twice the difference of their squares multiplied by the greater is equal to 15 times the less. Required the two numbers.

(24.) A farmer at a fair found the price of a cow equal to that of three sheep, and saw that he could just dispose of £100 in buying twice as many sheep as cows. But waiting till the evening, when the price of a cow

fell £1 and of a sheep 6s. 8d., he bought for £100 three times as many sheep as cows, and increased his whole stock by ten more than he would have done in the former case. How many sheep and cows did he buy, and what was the price of each?

CHAPTER XIII.

SURDS.

166. The definition of a surd will be found in Arts. 124 and 138.

167. Fractional Indices.—From Art. 139 we find that the root of a simple quantity is obtained by dividing the exponent of the given quantity by the number indicating the root required. Thus, the square root of a^4 is a^2 or $a^{\frac{4}{2}}$, and the fifth root of x^{15} is x^3 or $x^{\frac{15}{5}}$, and generally the nth root of a^m is $a^{\frac{m}{n}}$ when m is divisible by n.

Observe here that the index takes the form of a fraction, the numerator expressing the power to which the quantity is to be raised, and the denominator the root to be extracted; also, that the index is rendered integral by dividing the numerator by the denominator. When, however, this cannot be done, the index becomes a true fraction, and we have such a quantity as $a^{\frac{m}{n}}$ where m is not divisible by n. This expression must evidently have such a meaning attached to it as shall be consistent with the one it has when m is divisible by n; and if we say that $a^{\frac{m}{n}}$ signifies the nth root of a^m, we shall give it such a meaning.

We know that $a^{\frac{1}{2}}$, $a^{\frac{1}{3}}$, $a^{\frac{1}{6}}$ indicate respectively the second, third, and sixth roots of a^{12}, and we agree that $a^{\frac{10}{6}}$, $a^{\frac{10}{4}}$, $a^{\frac{10}{3}}$ shall represent the sixth, fourth, and third roots of a^{10}.

Similarly, the third root of a^2 may be indicated by $a^{\frac{2}{3}}$, and the fourth root of a by $a^{\frac{1}{4}}$; and as these may also be represented respectively by $\sqrt[3]{a^2}$ and $\sqrt[4]{a}$, we have $\sqrt[3]{a^2} = a^{\frac{2}{3}}$ and $\sqrt[4]{a} = a^{\frac{1}{4}}$, and generally $\sqrt[n]{a^m} = a^{\frac{m}{n}}$.

This is also true of compound quantities, so that $\sqrt[n]{(a+x)^m} = (a+x)^{\frac{m}{n}}$.

168. Let $a^m = x$, then $a^{\frac{m}{n}} = x^{\frac{1}{n}} = (a^m)^{\frac{1}{n}}$.

Again, let $a^{\frac{1}{n}} = x$, then $a^{\frac{m}{n}} = x^m = (a^{\frac{1}{n}})^m$.

From which we find that

$$a^{\frac{m}{n}} = (a^m)^{\frac{1}{n}} = (a^{\frac{1}{n}})^m,$$

that is, $a^{\frac{m}{n}}$ expresses the $\frac{1}{n}$th power of a^m, or mth power of $a^{\frac{1}{n}}$.

169. Surds under the same root-sign are said to have the same index, or to be of the same order.

$\sqrt{a}$, $\sqrt[3]{a}$, $\sqrt[4]{a}$, $\sqrt[n]{a}$ are respectively of the second, third, fourth, and nth order. Those of the second order are sometimes called *quadratic* surds; and those of the third, *cubic*.

Surds which consist of the same quantity under the same root-sign are said to be like.

170. Reduction of Surds.—Let it be required to find the product of $\sqrt{a}$ by $\sqrt{b}$.

Put $\sqrt{a} \times \sqrt{b} = x$

Squaring, then $a \times b = x^2$

or $ab = x^2$

Extracting root, $\sqrt{ab} = x$

That is, $\sqrt{a} \times \sqrt{b} = \sqrt{ab}$.

Similarly we may show that $\sqrt[3]{a} \times \sqrt[3]{b} = \sqrt[3]{ab}$, and generally that $\sqrt[n]{a} \times \sqrt[n]{b} = \sqrt[n]{ab}$.

In like manner also we may prove that

$$\frac{\sqrt[n]{a}}{\sqrt[n]{b}} = \sqrt[n]{\frac{a}{b}}$$

From this we have the following

171. Preliminary Proposition.—The product or quotient of two surds having the same index is a surd of the same order, the multiplication or division indicated being performed on the quantity under the root-signs, and the result placed under a root-sign of the same kind.

Illustrative Examples.

(1.) Multiply $\sqrt{a}$ by $\sqrt{x}$. *Ans.* $= \sqrt{ax}$.

(2.) Multiply $\sqrt{ab}$ by $\sqrt{a+b}$. *Ans.* $= \sqrt{ab(a+b)}$.

(3.) Divide $\sqrt{a}$ by $\sqrt{x}$. *Ans.* $= \sqrt{\frac{a}{x}}$

(4.) Divide $\sqrt{m-n}$ by $\sqrt{m+n}$. *Ans.* $= \sqrt{\frac{m-n}{m+n}}$.

Reduction naturally arranges itself into the following propositions:—

172. I. To bring a Rational Quantity to the form of a Surd.

Raise the given quantity to the power which is inverse to that of the required root, and place it under the proper root-sign.

Illustrative Examples.

(1.) Express $a+x$ as a quadratic surd.

$$a+x=\sqrt{(a+x)^2} \text{ or } \{(a+x)^2\}^{\frac{1}{2}}.$$

(2.) Reduce $3a$ to the form of a surd with the index $\frac{1}{4}$.

$$3a=\sqrt[4]{3^4a^4} \text{ or } (81a^4)^{\frac{1}{4}}.$$

173. II. To bring a Product, of which the Factors are partly Rational, partly Surd, entirely to the Surd form.

Raise the rational part to the power inverse to that expressed by the index of the surd, and multiply it into the quantity under the root-sign.

Illustrative Examples.

(1.) Express $2ab^2\sqrt[3]{x^2}$ entirely as a surd.

$$2ab^2\sqrt[3]{x^2} = \sqrt[3]{8a^3b^6}\times\sqrt[3]{x^2} \text{ or } (8a^3b^6)^{\frac{1}{3}}(x^2)^{\frac{1}{3}}$$
$$= \sqrt[3]{8a^3b^6x^2} \text{ or } (8a^3b^6x^2)^{\frac{1}{3}}$$

(2.) Reduce $\frac{3}{2}m\sqrt{\frac{1}{6m}(m+n)}$ to the form of an entire surd.

$$\frac{3}{2}m\sqrt{\frac{1}{6m}(m+n)}=\sqrt{\frac{9}{4}m^2}\sqrt{\frac{1}{6m}(m+n)}$$
$$=\sqrt{\frac{9m^2}{24m}(m+n)}=\sqrt{\frac{3}{8}m(m+n)}=\left\{\frac{3}{8}m(m+n)\right\}^{\frac{1}{2}}.$$

174. III. To bring to its simplest form a Surd which contains a Rational Factor or Factors.

Extract the root of the rational quantity, and let it form a coefficient to the remaining surd.

Illustrative Examples.

(1.) Bring the expression $\sqrt[4]{16a^6b^4x^3}$ to its simplest form.

$$\sqrt[4]{16a^6b^4x^3}=\sqrt[4]{2^4a^4b^4}\times a^2x^3=\sqrt[4]{2^4a^4b^4}\sqrt[4]{a^2x^3}$$
$$=2ab\sqrt[4]{a^2x^3} \text{ or } 2aba^{\frac{1}{2}}x^{\frac{3}{4}}.$$

(2.) Simplify $\sqrt[3]{\frac{2ab^4(a+b)^5}{3c^3(a-b)^2}}$.

$$\sqrt[3]{\frac{2ab^4(a+b)^5}{3c^3(a-b)^2}} = \sqrt[3]{\frac{b^3(a+b)^3 \times 2ab(a+b)^2}{c^3 \times 3(a-b)^2}}$$

$$= \frac{b(a+b)}{c}\sqrt[3]{\frac{2ab(a+b)^2}{3(a-b)^2}} \text{ or } \frac{b}{c}(a+b)\left\{\frac{2ab(a+b)^2}{3(a-b)^2}\right\}^{\frac{1}{3}}.$$

Corollary: Any quantity may be made a coefficient to a surd, provided that the expression under the root-sign is divided by that quantity raised to a power corresponding to the root indicated.

Illustrative Examples.

(1.) $\sqrt[3]{a^2bc} = a\sqrt[3]{\frac{a^2bc}{a^3}} = a\sqrt[3]{\frac{bc}{a}}$

Similarly, $\sqrt[3]{a^2bc} = b\sqrt[3]{\frac{a^2c}{b^2}} = c\sqrt[3]{\frac{a^2b}{c^2}} = m\sqrt[3]{\frac{a^2bc}{m^3}}$.

(2.) $\sqrt{a^2-b^2} = a\sqrt{1-\frac{b^2}{a^2}} = b\sqrt{\frac{a^2}{b^2}-1}$

$$= m\sqrt{\frac{a^2-b^2}{m^2}} = m\sqrt{(a^2-b^2)m^{-2}}.$$

175. IV. To change the Index of a Surd.

Bring the quantity under the root-sign to the power required, and indicate the extraction of the corresponding root.

Illustrative Examples.

(1.) Write $\sqrt[5]{a^4b^3x}$ with the index $\frac{1}{10}$.

$$\sqrt[5]{a^4b^3x} = (a^4b^3x)^{\frac{1}{5}} = (a^4b^3x)^{\frac{2}{10}}$$

$$= (a^8b^6x^2)^{\frac{1}{10}} = \sqrt[10]{a^8b^6x^2}.$$

(2.) What form does $\sqrt[3]{(m-n)^2}$ assume, when the exponent is $\frac{5}{6}$?

$$\sqrt[3]{(m-n)^2} = (m-n)^{\frac{2}{3}} = (m-n)^{\frac{2}{3}\cdot\frac{5}{6}\cdot\frac{6}{5}} = (m-n)^{\frac{5}{6}\cdot\frac{4}{5}}$$
$$= \{(m-n)^{\frac{5}{6}}\}^{\frac{4}{5}} \text{ or } \{\sqrt[5]{(m-n)^4}\}^{\frac{5}{6}}.$$

176. V. To bring Two or more Surds to a Common Index.

Reduce the indices of the given surds to fractions of the same denominator. The denominator will give the common index, and the numerators the powers to which the quantities under the root-sign must be raised.

Illustrative Examples.

(1.) Reduce $\sqrt{2ax}$ and $\sqrt[3]{3a^2x}$ to a common index.

The index of the first being $\frac{1}{2}$, and that of the second $\frac{1}{3}$, the common index is $\frac{1}{6}$.

Then $\sqrt{2ax} = (2ax)^{\frac{1}{2}} = (2ax)^{\frac{3}{6}} = \sqrt[6]{2^3a^3x^3}$

And $\sqrt[3]{3a^2x} = (3a^2x)^{\frac{1}{3}} = (3a^2x)^{\frac{2}{6}} = \sqrt[6]{3^2a^4x^2}$.

(2.) Reduce $\sqrt{a^2-b^2}$, $\sqrt[4]{a^2(a+b)}$, and $\sqrt[6]{b^2(a-b)}$ to the same index.

The least common denominator of $\frac{1}{2}$, $\frac{1}{4}$, and $\frac{1}{6}$ being 12, the common index is $\frac{1}{12}$.

Then $\sqrt{a^2-b^2} = (a^2-b^2)^{\frac{1}{2}} = (a^2-b^2)^{\frac{6}{12}} = \sqrt[12]{(a^2-b^2)^6}$

$\sqrt[4]{a^2(a+b)} = \{a^2(a+b)\}^{\frac{1}{4}} = \{a^2(a+b)\}^{\frac{3}{12}} = \sqrt[12]{a^6(a+b)^3}$

and $\sqrt[6]{b^2(a-b)} = \{b^2(a-b)\}^{\frac{1}{6}} = \{b^2(a-b)\}^{\frac{2}{12}} = \sqrt[12]{b^4(a-b)^2}$.

After the common index has been found, the transformation may be made at once, by multiplying each of the numbers indicating the roots by such a number as will bring it to the one required, and multiplying the exponents of

each of the factors under the root-signs by the same number.

Thus, $\sqrt[2]{(a^2-b^2)} = \sqrt[12]{(a^2-b^2)^6}$ multiplying exponents by 6.

$\sqrt[4]{a^2(a+b)} = \sqrt[12]{a^6(a+b)^3}$ „ „ 3.

$\sqrt[6]{b^2(a+b)} = \sqrt[12]{b^4(a+b)^2}$ „ „ 2.

General Illustrative Examples.

(1.) Express $\sqrt[6]{a^2b^3}$ and $\sqrt[3]{x^2(a^2-x^2)}$ with fractional indices.

By Art. 167, $\sqrt[6]{a^2b^3} = (a^2b^3)^{\frac{1}{6}} = a^{\frac{2}{6}}b^{\frac{3}{6}} = a^{\frac{1}{3}}b^{\frac{1}{2}}$

and $\sqrt[3]{x^2(a^2-x^2)} = x^{\frac{2}{3}}(a^2-x^2)^{\frac{1}{3}}$.

(2.) Reduce $5a^3x^2(a-x)$ to the form of a cubic surd.

By Art. 172, $5a^3x^2(a-x) = \sqrt[3]{125a^9x^6(a-x)^3}$.

(3.) Write $\frac{1}{2}m\sqrt[3]{4n^2}$ and $(x-a)\sqrt{\frac{x+a}{x-a}}$ entirely as surds.

By Art. 173, $\frac{1}{2}m\sqrt[3]{4n^2} = \sqrt[3]{\frac{4m^3n^2}{8}} = \sqrt[3]{\frac{1}{2}m^3n^2}$

and $(x-a)\sqrt{\frac{x+a}{x-a}} = \sqrt{\frac{(x-a)^2(x+a)}{x-a}} = \sqrt{x^2-a^2}$.

(4.) Reduce $\frac{1}{a}\sqrt{18a^7x - 36a^6x^2 + 18a^5x^3}$ to its simplest form.

By Art. 174,

$$\frac{1}{a}\sqrt{18a^7x - 36a^6x^2 + 18a^5x^3} = \frac{1}{a}\sqrt{18a^5x(a^2-2ax+x^2)}$$

$$= \frac{1}{a}\sqrt{9a^4(a-x)^2 \times 2ax} = \frac{3a^2(a-x)}{a}\sqrt{2ax}$$

$$= 3a(a-x)\sqrt{2ax}.$$

(5.) Write $\sqrt{ab(a^2+b^2)}$ with ab as a coefficient to the surd.

By Art. 174 (corollary), $\sqrt{ab(a^2+b^2)} = ab\sqrt{\frac{ab(a^2+b^2)}{a^2b^2}}$

$$= ab\sqrt{\frac{a^2+b^2}{ab}} = ab\sqrt{\frac{a}{b}+\frac{b}{a}}.$$

(6.) Change the index of $\sqrt[4]{a^2(a-x)^3}$ to $\frac{1}{6}$.

By Art. 175, $\sqrt[4]{a^2(a-x)^3} = \sqrt[6]{a^3(a-x)^3}$.

(7.) Reduce $\sqrt{ax}$, $\sqrt[3]{bx^2}$, and $\sqrt[5]{cx^3}$ to the same order.

The lowest common index is $\frac{1}{30}$.

By Art. 176, $\sqrt{ax} = \sqrt[30]{a^{15}x^{15}}$, $\sqrt[3]{bx^2} = \sqrt[30]{b^{10}x^{20}}$, and $\sqrt[5]{cx^3} = \sqrt[30]{c^6x^{18}}$.

EXAMPLES FOR PRACTICE.—LX.

(1.) Express $\sqrt{a}$, $\sqrt[3]{a^2x}$, $\sqrt[6]{4b^3x^2y}$, $\sqrt[4]{(a^2-x^2)x^3y^6}$, $\sqrt[12]{\frac{a^6b^4c^2}{x^5y^3z}}$, and $\sqrt[15]{\frac{(a-b)^{10}}{(a+b)^6}x^5}$ with fractional indices.

(2.) Write $a^{\frac{3}{4}}$, $a^{\frac{3}{4}}x^{\frac{1}{4}}$, $(a^2-1)^{\frac{1}{2}}x^{\frac{2}{3}}y^{\frac{1}{3}}$, and $\frac{3^{\frac{1}{2}}(x-y)^{\frac{3}{5}}}{(x^2+y^2)^{\frac{3}{5}}}$ under root-signs.

(3.) Express $2a^3m^2(m-a)$ as a cubic surd.

(4.) Reduce $\frac{a^3x^2(a^2-x^2)^{\frac{1}{2}}}{3mn^{\frac{1}{3}}}$ to the form of a surd of the sixth order.

(5.) Bring the coefficients under the root-sign, in $5\sqrt{xy}$, $-2a\sqrt[3]{a^2+b^2}$, $\frac{a}{b}\sqrt[4]{\frac{b}{a}}$, $\frac{1}{3}(a+b)\sqrt[5]{\frac{27(a-b)^2}{(a+b)^3}}$, and $\sqrt{ab}\sqrt[6]{\frac{1}{ab}(x^2-xy+y^2)}$.

(6.) Resolve into rational and irrational factors $\sqrt{112}$, $\sqrt[3]{81a^5b^2}$, $\sqrt[4]{64(a^2-x^2)^2(a-x)^2}$, $\sqrt{5a^3-10a^2x+5ax^2}$, $\sqrt[6]{\frac{b^{12}x^9}{a^6y^{14}}}$, and $\sqrt[3]{\left(\frac{a^4+x^4}{2x^2}-a^2\right)\left(\frac{a^2}{x}-x\right)}$.

(7.) Write $\sqrt[3]{4ax^2}$ with the coefficient 4, and $\sqrt{a^3-x^3}$ with the coefficient $a-x$.

(8.) Change $\sqrt{3(a+x)(b+y)}$ to a surd of the fourth order, and write $\sqrt[4]{a^3(x-y)^2(y-z)}$ with the fractional index $\frac{1}{8}$.

(9.) Reduce $\sqrt{8}$ and $\sqrt[3]{9}$ to the lowest common index; also $\sqrt[3]{a^2b}$, $\sqrt[6]{a^3(a-x)^2}$, and $\sqrt{a(b-x)}$ to the index $\frac{1}{12}$.

(10.) Simplify $\sqrt{a^3-ax^2}\sqrt{ax-x^2}$ and $\sqrt[3]{\frac{a^2xy}{bcz^2}} \div \sqrt[3]{\frac{b^2z}{acxy^2}}$.

(11.) Express $a^2x\sqrt[n]{ax^2}$ and $\frac{ab}{a-b}\sqrt[n]{\left(\frac{a-b}{ab}\right)^p}$ as entire surds, the latter with a fractional index, n being $> p$.

(12.) Write in simplest form, $a^{-1}\sqrt[4]{(a^2-x^2)^3(a^5+a^4x)}$ and $\sqrt[n]{a^{m+n}(a-x)^{2m-n}} \sqrt[n]{x^{m+n}(a-x)^{n-2m}}$.

177. Addition and Subtraction. — It is evident that addition and subtraction of surds must be performed in the same manner as in the case of ordinary quantities.

RULE.—When the surds are like, add or subtract their coefficients as may be required; when the surds are unlike, indicate their addition or subtraction by employing the signs + or −.

Before adding or subtracting, reduce the surds to their simplest form (Art. 174).

Illustrative Examples.

(1.) Find the sum of $3x\sqrt{8a^3x}$, $2a\sqrt{2ax^3}$, and $ax\sqrt{32ax}$.

$$3x\sqrt{8a^3x} = 3x\sqrt{4a^2 \times 2ax} = 6ax\sqrt{2ax}$$
$$2a\sqrt{2ax^3} = 2a\sqrt{x^2 \times 2ax} = 2ax\sqrt{2ax}$$
$$ax\sqrt{32ax} = ax\sqrt{16 \times 2ax} = 4ax\sqrt{2ax}.$$

As the root-signs and the quantities under them are the same, the surds are like, and their sum will be found by adding their coefficients; this gives—

$$12ax\sqrt{2ax}.$$

$$\therefore\ 3x\sqrt{8a^3x} + 2a\sqrt{2ax^3} + ax\sqrt{32ax} = 12ax\sqrt{2ax}.$$

(2.) Subtract $3x\sqrt[3]{a^3x^2y^2} - \sqrt{a^3x^3}$ from $4a\sqrt[3]{x^5y^2} - \sqrt{a^3y^3}$.

Simplifying, $4a\sqrt[3]{x^5y^2} - \sqrt{a^3y^3} = 4ax\sqrt[3]{x^2y^2} - ay\sqrt{ay}$

And $3x\sqrt[3]{a^3x^2y^2} - \sqrt{a^3x^3} = 3ax\sqrt[3]{x^2y^2} - ax\sqrt{ax}$.

Here the first terms are like and the second unlike; the difference will therefore be $ax\sqrt[3]{x^2y^2} + ax\sqrt{ax} - ay\sqrt{ay}$.

(3.) Reduce to as simple a form as possible,

$$\sqrt[4]{4a^2x^2} + 2\sqrt[3]{4a^4x} - \sqrt{8a^3x} - \sqrt[3]{8a^4x^2} + \sqrt{2a^5x}.$$

Simplifying, this becomes—

$$\sqrt{2ax} + 2a\sqrt[3]{4ax} - 2a\sqrt{2ax} - 2a\sqrt[3]{ax^2} + a^2\sqrt{2ax}.$$

Collecting, $(1 - 2a + a^2)\sqrt{2ax} + 2a(\sqrt[3]{4ax} - \sqrt[3]{ax^2})$

or $(1-a)^2\sqrt{2ax} + 2a(\sqrt[3]{4ax} - \sqrt[3]{ax^2})$.

EXAMPLES FOR PRACTICE.—LXI.

(1.) Find the sum of $\sqrt{2a}$, $\sqrt{8a}$, and $\sqrt{32a}$.

(2.) Collect $\sqrt[3]{81x^4} + \sqrt[3]{24xy^3} - \sqrt[3]{3x^4y^3}$.

(3.) Add together—

$$\sqrt{a^3 + a^2x},\ \sqrt{ax^2 + x^3},\ \text{and}\ \sqrt{(a^2 - x^2)(a - x)}.$$

(4.) Subtract $\sqrt[4]{1536xy}$ from $2\sqrt[4]{486xy}$.

(5.) By what does $\sqrt{27a^3x - 108a^2x^2 + 108ax^3}$ exceed $\sqrt{12a^3x - 72a^2x^2 + 108ax^3}$?

(6.) Simplify $a(a+x)^{\frac{m}{n}} - \sqrt[n]{(a+x)^{m+n}} + x(a+x)^{\frac{m}{n}}$.

178. Multiplication and Division.—In Art. 170 it was shown that $\sqrt[n]{a} \times \sqrt[n]{b} = \sqrt[n]{ab}$, and similarly we can prove that $\sqrt[n]{a} \times \sqrt[n]{b} \times \sqrt[n]{c} = \sqrt[n]{abc}$, and so on for any number of quantities.

From the same article it also appears that $\frac{\sqrt[n]{a}}{\sqrt[n]{b}} = \sqrt[n]{\frac{a}{b}}$. Consequently the multiplication and the division of surds of the *same order* are performed as there indicated.

The same method may also be applied to surds of *different orders*, for by Art. 176 such surds may be reduced to a common index.

We have therefore the following

GENERAL RULE.—Reduce the surds to the same order if necessary: perform upon the quantities under the several root-signs the multiplication or the division indicated, and place the result under the common root-sign. Rational coefficients must also be multiplied or divided.

Illustrative Examples.

(1.) Multiply together $\sqrt[3]{a^2b}$ and $\sqrt[3]{bx^2}$.

$$\sqrt[3]{a^2b} \times \sqrt[3]{bx^2} = \sqrt[3]{a^2b \times bx^2} = \sqrt[3]{a^2b^2x^2}.$$

(2.) Find the product of $\sqrt{ax}$ and $\sqrt[3]{a^2bx}$.

By Art. 170, $\sqrt{ax} = \sqrt[6]{a^3x^3}$ and $\sqrt[3]{a^2bx} = \sqrt[6]{a^4b^2x^2}$.

$$\therefore \sqrt{ax} \times \sqrt[3]{a^2bx} = \sqrt[6]{a^3x^3} \times \sqrt[6]{a^4b^2x^2}$$
$$= \sqrt[6]{a^7b^2x^5} = a\sqrt[6]{ab^2x^5}.$$

(3.) Divide $\sqrt[4]{a^2b^3x^2}$ by $\sqrt[3]{a^2b^2cx}$.

By Art. 170, $\dfrac{\sqrt[4]{a^2b^3x^2}}{\sqrt[3]{a^2b^2cx}} = \dfrac{\sqrt[12]{a^6b^9x^6}}{\sqrt[12]{a^8b^8c^4x^4}} = \sqrt[12]{\dfrac{bx^2}{a^2c^4}}$.

(4.) Simplify $\dfrac{\sqrt{ax(a^2-x^2)}\,\sqrt[3]{a^2x^2(a^2+ax+x^2)}}{\sqrt{a^3-x^3}\,\sqrt[3]{a+x}}$.

$$\frac{\sqrt{ax(a^2-x^2)}\,\sqrt[3]{a^2x^2(a^2+ax+x^2)}}{\sqrt{a^3-x^3}\,\sqrt[3]{a+x}}$$

$$= \frac{\sqrt[6]{a^3x^3(a^2-x^2)^3}\,\sqrt[6]{a^4x^4(a^2+ax+x^2)^2}}{\sqrt[6]{(a^3-x^3)^3}\,\sqrt[6]{(a+x)^2}}$$

$$= \sqrt[6]{\frac{a^7x^7(a^2-x^2)^3(a^2+ax+x^2)^2}{(a^3-x^3)^3(a+x)^2}}$$

$$= \sqrt[6]{\frac{a^7x^7(a-x)^3(a+x)^3(a^2+ax+x^2)^2}{(a-x)^3(a^2+ax+x^2)^3(a+x)^2}}$$

$$= ax\sqrt[6]{\frac{ax(a+x)}{a^2+ax+x^2}}.$$

179. When the literal parts of the surds are the same, the multiplication or the division may frequently be more conveniently performed by employing fractional indices.

By Art. 167, $\sqrt[n]{a^m} = a^{\frac{m}{n}}$ and $\sqrt[q]{a^p} = a^{\frac{p}{q}}$.

$$\therefore \sqrt[n]{a^m} \times \sqrt[q]{a^p} = a^{\frac{m}{n}} \times a^{\frac{p}{q}}.$$

But $\sqrt[n]{a^m} \times \sqrt[q]{a^p} = \sqrt[nq]{a^{mq}} \times \sqrt[nq]{a^{np}}$, By Art. 176.

$= \sqrt[nq]{a^{mq+np}}$, By Art. 178.

$= a^{\frac{mq+np}{nq}}$, By Art. 167.

$= a^{\frac{m}{n}+\frac{p}{q}}$.

$$\therefore a^{\frac{m}{n}} \times a^{\frac{p}{q}} = a^{\frac{m}{n}+\frac{p}{q}}.$$

Similarly we can show that—

$$a^{\frac{m}{n}} \div a^{\frac{p}{q}} = a^{\frac{m}{n} - \frac{p}{q}}.$$

From this it appears that the product of two fractional powers of the same quantity is obtained by adding the fractional exponents together, and the quotient by subtracting the exponent of the divisor from that of the dividend, as in the case of integral indices (Art. 50).

Illustrative Examples.

(1.) Multiply $a^{\frac{1}{2}}b^{\frac{1}{3}}x^{\frac{1}{4}}$ by $a^{\frac{1}{3}}b^{\frac{1}{4}}x^{\frac{1}{5}}$.

$$a^{\frac{1}{2}}b^{\frac{1}{3}}x^{\frac{1}{4}} \times a^{\frac{1}{3}}b^{\frac{1}{4}}x^{\frac{1}{5}} = a^{\frac{1}{2}+\frac{1}{3}}\, b^{\frac{1}{3}+\frac{1}{4}}\, x^{\frac{1}{4}+\frac{1}{5}}$$

$$= a^{\frac{5}{6}}b^{1\frac{7}{12}}x^{\frac{9}{20}} \qquad [A]$$

$$= \sqrt[60]{a^{50}b^{35}x^{27}}, \qquad \text{By Art. 176.}$$

In most cases it will suffice to leave the answer in the fractional form, as at [A].

(2.) Divide $\sqrt[6]{a^3b^2c^3x^4}$ by $\sqrt[4]{ab^2c^2x^3}$.

$$\frac{\sqrt[6]{a^3b^2c^3x^4}}{\sqrt[4]{ab^2c^2x^3}} = \frac{a^{\frac{1}{2}}b^{\frac{1}{3}}c^{\frac{1}{2}}x^{\frac{2}{3}}}{a^{\frac{1}{4}}b^{\frac{1}{2}}c^{\frac{1}{2}}x^{\frac{3}{4}}},$$

$$= a^{\frac{1}{4}}b^{-\frac{1}{6}}x^{-1\frac{1}{12}} \text{ or } \frac{a^{\frac{1}{4}}}{b^{\frac{1}{6}}x^{1\frac{1}{12}}}.$$

EXAMPLES FOR PRACTICE.—LXII.

(1.) Multiply $\sqrt{ax}$ by $\sqrt{bx}$, and $\sqrt{ax}$ by $\sqrt[3]{b^2x^2}$.

(2.) Multiply $\sqrt[4]{28}$ by $\sqrt{24}$ and $4\sqrt[4]{4a}$ by $3\sqrt[3]{4a^2}$.

(3.) Find the product of $\sqrt{a^3-b^3}$, $\sqrt[3]{a^2-b^2}$, and $\sqrt[4]{a-b}$.

(4.) Divide $4\sqrt{ax}$ by $8\sqrt{bx}$ and $6\sqrt{\frac{x}{y}}$ by $8\sqrt{\frac{y}{x}}$.

(5.) Divide $a^2xy^{\frac{1}{2}}$ by $a^{\frac{1}{2}}x^{\frac{1}{3}}y^{\frac{2}{3}}$, and $m^{\frac{1}{6}}n^{\frac{1}{3}}x^{\frac{1}{2}}$ by $(m^4n^2x^5)^{\frac{1}{12}}$.

(6.) Simplify $\sqrt{a\sqrt{a\sqrt{a}}}$, $\sqrt{x^{\frac{1}{2}}x^{\frac{2}{3}}x^{\frac{4}{5}}\sqrt{x-y}}$,

$\sqrt[4]{a^{\frac{m+n}{m-n}} \div a^{\frac{n-m}{m+n}}}$, and $\dfrac{(ax-x^2)^{\frac{1}{m}}(a^2-ax)^{\frac{1}{n}}}{(a^2-x^2)^{\frac{m+n}{mn}}}$.

180. Compound Surds.—To such expressions as $2-\sqrt{3}$, $\sqrt[3]{ax}+\sqrt{by}$ and $a^{\frac{2}{3}}-a^{\frac{1}{3}}+1$, which are spoken of as compound surds, we are now able to apply all the ordinary rules of addition, subtraction, multiplication, and division that are applicable to integral quantities.

Illustrative Examples.

Examples in addition and subtraction have already been given (Art. 177).

(1.) Multiply $3\sqrt{x}-2\sqrt{y}$ by $2\sqrt{x}+3\sqrt{y}$.

$$\begin{array}{r} 3\sqrt{x}-2\sqrt{y} \\ 2\sqrt{x}+3\sqrt{y} \\ \hline 6x-4\sqrt{xy} \\ \quad +9\sqrt{xy}-6y \\ \hline 6x+5\sqrt{xy}-6y \end{array} = \text{product.}$$

(2.) Multiply $x^{\frac{3}{4}}-2x^{\frac{1}{2}}+3x^{\frac{1}{4}}-1$ by $x^{\frac{1}{2}}-2$.

$$\begin{array}{l} x^{\frac{3}{4}}-2x^{\frac{1}{2}}+3x^{\frac{1}{4}}-1 \\ x^{\frac{1}{2}}-2 \\ \hline x^{\frac{5}{4}}-2x\ +3x^{\frac{3}{4}}-x^{\frac{1}{2}} \\ \qquad -2x^{\frac{3}{4}}+4x^{\frac{1}{2}}-6x^{\frac{1}{4}}+2 \\ \hline x^{\frac{5}{4}}-2x\ +\ x^{\frac{3}{4}}+3x^{\frac{1}{2}}-6x^{\frac{1}{4}}+2. \end{array}$$

(3.) Divide $a^m+a^{\frac{1}{2}m}x^{\frac{1}{2}n}+x^n$ by $a^{\frac{1}{2}m}-a^{\frac{1}{4}m}x^{\frac{1}{4}n}+x^{\frac{1}{2}n}$.

$$a^{\frac{1}{2}m}-a^{\frac{1}{4}m}x^{\frac{1}{4}n}+x^{\frac{1}{2}n}\;\Big)\,a^{m}+a^{\frac{1}{2}m}x^{\frac{1}{2}n}+x^{n}\;\Big(\;a^{\frac{1}{2}m}+a^{\frac{1}{4}m}x^{\frac{1}{4}n}+x^{\frac{1}{2}n}$$
$$a^{m}-a^{\frac{3}{4}m}x^{\frac{1}{4}n}+a^{\frac{1}{2}m}x^{\frac{1}{2}n}$$
$$a^{\frac{3}{4}m}x^{\frac{1}{4}n}+x^{n}$$
$$a^{\frac{3}{4}m}x^{\frac{1}{4}n}-a^{\frac{1}{2}m}x^{\frac{1}{2}n}+a^{\frac{1}{4}m}x^{\frac{3}{4}n}$$
$$a^{\frac{1}{2}m}x^{\frac{1}{2}n}-a^{\frac{1}{4}m}x^{\frac{3}{4}n}+x^{n}$$
$$a^{\frac{1}{2}m}x^{\frac{1}{2}n}-a^{\frac{1}{4}m}x^{\frac{3}{4}n}+x^{n}.$$

(4.) Raise $a^{\frac{1}{4}}-x^{\frac{1}{2}}$ to the fourth power.

This may be done by actual multiplication, or by applying the method of Art. 134, thus—

$$(a^{\frac{1}{4}}-x^{\frac{1}{2}})^4=(a^{\frac{1}{4}})^4-4(a^{\frac{1}{4}})^3(x^{\frac{1}{2}})+6(a^{\frac{1}{4}})^2(x^{\frac{1}{2}})^2-4(a^{\frac{1}{4}})(x^{\frac{1}{2}})^3+(x^{\frac{1}{2}})^4$$
$$=a-4a^{\frac{3}{4}}x^{\frac{1}{2}}+6a^{\frac{1}{2}}x-4a^{\frac{1}{4}}x^{\frac{3}{2}}+x^2.$$

EXAMPLES FOR PRACTICE.—LXIII.

(1.) Add $\sqrt{a}+\sqrt[3]{b}-\sqrt[4]{c}$, $\sqrt{4a}-4\sqrt[3]{b}+5\sqrt[4]{c}$, $3\sqrt{a}-\sqrt[3]{27b}-3\sqrt[4]{2c}$, and $\sqrt[3]{8b}-\sqrt[4]{16c}+2\sqrt[4]{2c}$.

(2.) Subtract $4\sqrt{xy}-2bx-\sqrt{9y}$ from $7\sqrt{xy}-4ax+5\sqrt{x}$.

(3.) Multiply $\sqrt{a}+\sqrt{b}$ by $\sqrt{a}-\sqrt{b}$.

(4.) Multiply $a^{\frac{2}{3}}-a^{\frac{1}{3}}b^{\frac{1}{3}}+b^{\frac{2}{3}}$ by $a^{\frac{1}{3}}+b^{\frac{1}{3}}$.

(5.) Multiply $a^{\frac{2}{3}}+a^{\frac{1}{3}}b^{\frac{1}{3}}+b^{\frac{2}{3}}$ by $a^{\frac{1}{3}}-b^{\frac{1}{3}}$.

(6.) Multiply $a+a^{\frac{1}{2}}b^{\frac{1}{2}}+b$ by $a-a^{\frac{1}{2}}b^{\frac{1}{2}}+b$.

(7.) Multiply $\sqrt{x}+\sqrt{xy}+\sqrt{y}$ by $\sqrt{x}-\sqrt{xy}+\sqrt{y}$.

(8.) Divide $x-3x^{\frac{2}{3}}+3x^{\frac{1}{3}}-1$ by $x^{\frac{1}{3}}-1$.

(9.) Divide $m-9m^{\frac{1}{2}}p^{-1}+12m^{\frac{1}{4}}p^{-\frac{3}{2}}-4p^{-2}$ by $m^{\frac{1}{2}}-3m^{\frac{1}{4}}p^{-\frac{1}{2}}+2p^{-1}$.

(10.) Find the continued product of $\dfrac{\sqrt{15}+2\sqrt{3}}{\sqrt{2}-1}$, $\dfrac{\sqrt{3}-1}{3\sqrt{5}+\sqrt{15}}$, and $\dfrac{\sqrt{10}-\sqrt{5}}{\sqrt{5}+2}$.

(11.) Find the quotient of $a^{\frac{4}{3}} - a^{-\frac{1}{3}}x^{\frac{4}{3}} - a^{-\frac{4}{3}}x^2 + a^{-\frac{5}{3}}x^{\frac{7}{3}}$ by $a^{\frac{1}{3}}x^{\frac{1}{3}} - a^{-1}x^{\frac{4}{3}}$.

(12.) Expand $(2\sqrt[3]{x} - \sqrt{x})^6$.

181. Rational Product of Surd Factors.—Observe in Nos. 3, 4, 5, and 6 of the above examples that we have the following results:—

3. $(\sqrt{a} + \sqrt{b})(\sqrt{a} - \sqrt{b}) = a - b$;
4. $(a^{\frac{2}{3}} - a^{\frac{1}{3}}b^{\frac{1}{3}} + b^{\frac{2}{3}})(a^{\frac{1}{3}} + b^{\frac{1}{3}}) = a + b$;
5. $(a^{\frac{2}{3}} + a^{\frac{1}{3}}b^{\frac{1}{3}} + b^{\frac{2}{3}})(a^{\frac{1}{3}} - b^{\frac{1}{3}}) = a - b$;
6. $(a + a^{\frac{1}{2}}b^{\frac{1}{2}} + b)(a - a^{\frac{1}{2}}b^{\frac{1}{2}} + b) = a^2 + ab + b^2$;

corresponding to Theorems III., IV., V., and VI. of Art. 57.

From this it appears that the product of certain compound surds is a rational quantity, and conversely that such expressions as $a+b$, $a-b$, can be resolved into factors.

182. In Art. 81 it was shown that when n is even, $x^n - a^n$ is always divisible by both $x+a$ and $x-a$; also, that when n is odd, $x^n + a^n$ is divisible by $x+a$, and $x^n - a^n$ by $x-a$. If we give the two terms of the above divisors fractional exponents, say $\frac{1}{p}$ and $\frac{1}{q}$, and let n be the L.C.M. of p and q, it can also be shown that when n is even, $x^{\frac{n}{p}} - a^{\frac{n}{q}}$ is divisible by both $x^{\frac{1}{p}} + a^{\frac{1}{q}}$ and $x^{\frac{1}{p}} - a^{\frac{1}{q}}$; also, that when n is odd, $x^{\frac{n}{p}} + x^{\frac{n}{q}}$ is divisible by $x^{\frac{1}{p}} + a^{\frac{1}{q}}$ and $x^{\frac{n}{p}} - a^{\frac{n}{q}}$ by $x^{\frac{1}{p}} - a^{\frac{1}{q}}$. So that we again have the product of two compound surds equal to a rational quantity.

It follows from this that for every binomial surd, as

well as some others of a greater number of terms, a multiplier may be found that will render it rational.

Illustrative Examples.

(1.) Resolve a^2-b, $2-m$, and $8+x$ into factors.

By Art. 182, $a^2-b=(a+\sqrt{b})(a-\sqrt{b})$.

$$2-m=(\sqrt{2}+\sqrt{m})(\sqrt{2}-\sqrt{m}).$$

$$8+x=(2+\sqrt[3]{x})(4-2\sqrt{x}+\sqrt[3]{x^2}).$$

The same quantity may frequently be resolved into surd factors in various ways; thus a^2-b may also become $(a+\sqrt{b})(\sqrt{a}+\sqrt[4]{b})(\sqrt{a}-\sqrt[4]{b})$ and $(a^{\frac{3}{3}}-b^{\frac{1}{3}})(a^{\frac{4}{3}}+a^{\frac{3}{3}}b^{\frac{1}{3}}+b^{\frac{2}{3}})$, &c.

(2.) Find the multiplier that will render $a^{\frac{1}{3}}+2^{\frac{1}{2}}$ rational.

Since 6 is the L.C.M. of 3 and 2, we must employ the formula $\frac{x^{\frac{n}{p}}-a^{\frac{n}{q}}}{x^{\frac{1}{p}}+a^{\frac{1}{q}}}$, which on substitution gives $\frac{a^{\frac{6}{3}}-2^{\frac{6}{2}}}{a^{\frac{1}{3}}+2^{\frac{1}{2}}}=\frac{a^2-2^3}{a^{\frac{1}{3}}+2^{\frac{1}{2}}}$

$=a^{\frac{5}{3}}-2^{\frac{1}{2}}a^{\frac{4}{3}}+2a-2^{\frac{3}{2}}a^{\frac{2}{3}}+2^2a^{\frac{1}{3}}-2^{\frac{5}{2}}$, the multiplier required.

(3.) Rationalize $\sqrt{a+x}+\sqrt{a-x}$.

Multiplying by $\sqrt{a+x}-\sqrt{a-x}$, we have—

$$\{\sqrt{a+x}+\sqrt{a-x}\}\{\sqrt{a+x}-\sqrt{a-x}\}=(a+x)-(a-x)=2x.$$

(4.) Rationalize $\sqrt{a}-\sqrt{b}+\sqrt{c}$.

Multiplying by $\sqrt{a}+\sqrt{b}-\sqrt{c}$, we have $a-b-c+2\sqrt{bc}$.

Multiplying this answer by $a-b-c-2\sqrt{bc}$, we have $(a-b-c)^2-4bc$, or $a^2+b^2+c^2-2(ab+ac+bc)$.

183. Rationalizing Denominator of a Fraction.— When the denominator of a fraction contains a surd or surds, it is frequently a matter of considerable convenience so to change the fraction as to remove the surd. In many cases this cannot be done, but whenever it is possible, the method of doing it consists in finding a suitable

multiplier (Art. 182), and using it to multiply all the terms of the fraction.

Illustrative Examples.

(1.) Express $\frac{1}{\sqrt{3}}$, $\frac{1}{\sqrt{2}+1}$, and $\frac{2}{\sqrt{7}-\sqrt{3}}$ with rational denominators.

$$\frac{1}{\sqrt{3}} = \frac{1}{\sqrt{3}} \times \frac{\sqrt{3}}{\sqrt{3}} = \frac{\sqrt{3}}{3} \text{ or } \frac{1}{3}\sqrt{3}.$$

$$\frac{1}{\sqrt{2}+1} = \frac{1}{\sqrt{2}+1} \times \frac{\sqrt{2}-1}{\sqrt{2}-1} = \frac{\sqrt{2}-1}{2-1} = \sqrt{2}-1.$$

$$\frac{2}{\sqrt{7}-\sqrt{3}} = \frac{2}{\sqrt{7}-\sqrt{3}} \times \frac{\sqrt{7}+\sqrt{3}}{\sqrt{7}+\sqrt{3}} = \frac{2(\sqrt{7}+\sqrt{3})}{7-3} = \tfrac{1}{2}(\sqrt{7}+\sqrt{3})$$

(2.) Find the sum of $\frac{1}{\sqrt{3}-\sqrt{2}+1}$ and $\frac{1}{\sqrt{3}-\sqrt{2}-1}$ in its simplest form.

$$\frac{1}{\sqrt{3}-\sqrt{2}+1} + \frac{1}{\sqrt{3}-\sqrt{2}-1} = \frac{\sqrt{3}-\sqrt{2}-1+\sqrt{3}-\sqrt{2}+1}{(\sqrt{3}-\sqrt{2})^2-1}$$

$$= \frac{2(\sqrt{3}-\sqrt{2})}{4-2\sqrt{6}} = \frac{\sqrt{3}-\sqrt{2}}{2-\sqrt{6}} = \frac{-(\sqrt{2}-\sqrt{3})}{\sqrt{2}(\sqrt{2}-\sqrt{3})}$$

$$= \frac{-1}{\sqrt{2}} = -\frac{1}{2}\sqrt{2}.$$

(3.) Rationalize the denominators of $\frac{1}{a^{\frac{1}{8}}-x^{\frac{1}{2}}}$ and $\frac{1}{a+x^{\frac{1}{8}}}$.

The L.C.M. of 8 and 2 is 8.

and $\frac{(a^{\frac{1}{8}})^8-(x^{\frac{1}{2}})^8}{a^{\frac{1}{8}}-x^{\frac{1}{2}}} = a^{\frac{7}{8}}+a^{\frac{3}{4}}x^{\frac{1}{2}}+a^{\frac{5}{8}}x+a^{\frac{1}{2}}x^{\frac{3}{2}}+a^{\frac{3}{8}}x^2+a^{\frac{1}{4}}x^{\frac{5}{2}}+a^{\frac{1}{8}}x^3+x^{\frac{7}{2}}.$

$$\therefore \frac{1}{a^{\frac{1}{8}}-x^{\frac{1}{2}}} = \frac{a^{\frac{7}{8}}+a^{\frac{3}{4}}x^{\frac{1}{2}}+a^{\frac{5}{8}}x+a^{\frac{1}{2}}x^{\frac{3}{2}}+a^{\frac{3}{8}}x^2+a^{\frac{1}{4}}x^{\frac{5}{2}}+a^{\frac{1}{8}}x^3+x^{\frac{7}{2}}}{a-x^4}$$

Similarly, $\frac{1}{a+x^{\frac{1}{8}}} = \frac{a^4-a^3x^{\frac{1}{8}}+a^2x^{\frac{2}{8}}-ax^{\frac{3}{8}}+x^{\frac{4}{8}}}{a^5+x}$.

EXAMPLES FOR PRACTICE.—LXIV.

(1.) Express $\frac{2}{\sqrt{5}}$, $\frac{3}{\sqrt[3]{4}}$, and $\frac{2}{\sqrt[4]{27}}$ with rational denominators.

(2.) Rationalize the denominators of $\frac{2}{\sqrt{11}-3}$ and $\frac{\sqrt{2}+\sqrt{5}}{\sqrt{8}+\sqrt{5}}$.

(3.) Reduce $\frac{1}{\sqrt[3]{a^4x^2}}$ and $\frac{\sqrt{a(a^2-x^2)}}{\sqrt[3]{a(a-x)^2}}$ to their simplest forms, with rational denominators.

(4.) Find the sum of $\frac{1+\sqrt{1-x}}{1+\sqrt{1+x}}$ and $\frac{1-\sqrt{1+x}}{1-\sqrt{1-x}}$, and take $\frac{1-\sqrt{3}}{1+\sqrt{3}}$ from $\frac{1-\sqrt{2}+\sqrt{3}}{1+\sqrt{2}+\sqrt{3}}$.

(5.) Rationalize the denominators of $\frac{a^{\frac{1}{2}}-x^{\frac{1}{2}}}{a-a^{\frac{1}{2}}x^{\frac{1}{2}}+x}$ and $\frac{1}{a^{\frac{3}{4}}-a^{\frac{1}{2}}x^{\frac{1}{4}}+a^{\frac{1}{4}}x^{\frac{1}{2}}-x^{\frac{3}{4}}}$.

(6.) Simplify

$$\frac{\sqrt{a^2-1}+\sqrt{a^2+1}}{\sqrt{a^2-1}-\sqrt{a^2+1}}+\frac{\sqrt{a+1}-\sqrt{a}}{\sqrt{a+1}+\sqrt{a}}+\frac{1}{a^2-\sqrt{a^4-1}}.$$

184. Theorems in Surds.—It will be necessary here to prove the following propositions :—

I. *The product of two unlike quadratic surds cannot be rational.*

Let $\sqrt{a}$ and $\sqrt{b}$ be two unlike surds, and, if possible, let $\sqrt{a}\times\sqrt{b}=m$, a rational quantity, then

$$ab=m^2, \text{ and } a=\frac{m^2}{b}=\frac{m^2}{b^2}b;$$

wherefore $\sqrt{a} = \frac{m}{b}\sqrt{b}$, that is, the one surd is a rational multiple of the other, from which it follows that they must be like, which is contrary to the supposition.

II. *A quadratic surd cannot be equal to the sum or the difference of a rational quantity and a quadratic surd, or of two quadratic surds.*

If possible, let $\sqrt{a} = m \pm \sqrt{b}$; then

Squaring, $a = m^2 \pm 2m\sqrt{b} + b$,

From which $\sqrt{b} = \pm \frac{a - b - m^2}{2m}$, that is, a surd is equal to a rational quantity, which is impossible.

Also, if we suppose $\sqrt{a} = \sqrt{b} \pm \sqrt{c}$, it can similarly be shown that $\sqrt{bc} = \pm \frac{1}{2}(a - b - c)$; that is, $\sqrt{bc}$ is a rational quantity, which by Theorem I. is impossible.

III. *If two expressions partly rational, and each containing a quadratic surd, be equal to one another, the rational part of the one must be equal to the rational part of the other, and the surd in the one to the surd in the other.*

Let $m + \sqrt{a} = n + \sqrt{b}$.

Then $m = n$ and $\sqrt{a} = \sqrt{b}$.

For if m be not equal to n, one of them must be the greater: let m be the greater, then $\sqrt{b}$ must be greater than $\sqrt{a}$, or $\sqrt{b}$ must be equal to the sum of a surd and a rational quantity, or of two surds, which is impossible by Theorem II. We have therefore $m = n$, and consequently $\sqrt{a} = \sqrt{b}$.

185. Extraction of Roots of Surd Quantities. — The roots of both simple and compound surds may generally be found by writing them with fractional indices, and applying the methods of Arts. 139 and 141.

Illustrative Example.

Find the square root of

$$a - 6a^{\frac{3}{4}}x^{\frac{1}{8}} + 13a^{\frac{1}{2}}x^{\frac{1}{4}} - 12a^{\frac{1}{4}}x^{\frac{3}{8}} + 4x^{\frac{1}{2}}.$$

$$\begin{array}{l|l|l}
a^{\frac{1}{2}} & a - 6a^{\frac{3}{4}}x^{\frac{1}{8}} + 13a^{\frac{1}{2}}x^{\frac{1}{4}} - 12a^{\frac{1}{4}}x^{\frac{3}{8}} + 4x^{\frac{1}{2}} & a^{\frac{1}{2}} \\
a^{\frac{1}{2}} & a & -3a^{\frac{1}{4}}x^{\frac{1}{8}} \\
\hline
2a^{\frac{1}{2}} - 3a^{\frac{1}{4}}x^{\frac{1}{8}} & -6a^{\frac{3}{4}}x^{\frac{1}{8}} + 13a^{\frac{1}{2}}x^{\frac{1}{4}} & +2x^{\frac{1}{4}} \\
\quad -3a^{\frac{1}{4}}x^{\frac{1}{8}} & -6a^{\frac{3}{4}}x^{\frac{1}{8}} + 9a^{\frac{1}{2}}x^{\frac{1}{4}} & \\
\hline
2a^{\frac{1}{2}} - 6a^{\frac{1}{4}}x^{\frac{1}{8}} + 2x^{\frac{1}{4}} & 4a^{\frac{1}{2}}x^{\frac{1}{4}} - 12a^{\frac{1}{4}}x^{\frac{3}{8}} + 4x^{\frac{1}{2}} & \\
 & 4a^{\frac{1}{2}}x^{\frac{1}{4}} - 12a^{\frac{1}{4}}x^{\frac{3}{8}} + 4x^{\frac{1}{2}} &
\end{array}$$

186. The Square Root of a Binomial Surd Quantity.— When a surd quantity of the form $\sqrt{a} + \sqrt{b}$ is raised to the second power, we have $a + 2\sqrt{ab} + b$, the square root of which may be easily extracted by the method of last example or by inspection.

When, however, the expressions $\sqrt{x+a} + \sqrt{x-a}$ and $\sqrt{7} - \sqrt{2}$ are raised to the second power and the like quantities added, we have $2x + 2\sqrt{x^2 - a^2}$ and $9 - 2\sqrt{14}$, binomial surds of the form $a + \sqrt{b}$, to which the above method obviously does not apply.

Let it be required to find the square root of $a + \sqrt{b}$.

Put	$\sqrt{x} + \sqrt{y} = \sqrt{a + \sqrt{b}}$	[1]
Squaring,	$x + y + 2\sqrt{xy} = a + \sqrt{b}$	[2]
By Theorem III., Art. 184,	$x + y = a$	[3]
	and $2\sqrt{xy} = \sqrt{b}$	[4]
[3] − [4],	$x - 2\sqrt{xy} + y = a - \sqrt{b}$	[5]
Extracting root,	$\sqrt{x} - \sqrt{y} = \sqrt{a - \sqrt{b}}$	[6]
[1] × [6],	$x - y = \sqrt{a^2 - b}$	[7]

From [3] and [7], $x = \frac{1}{2}(a + \sqrt{a^2 - b})$

and $y = \frac{1}{2}(a - \sqrt{a^2 - b})$

So that $\sqrt{a + \sqrt{b}} = \sqrt{x} + \sqrt{y}$

$$= \sqrt{\tfrac{1}{2}(a + \sqrt{a^2 - b})} + \sqrt{\tfrac{1}{2}(a - \sqrt{a^2 - b})}$$

From line [6] it also appears that

$$\sqrt{a - \sqrt{b}} = \sqrt{\tfrac{1}{2}(a + \sqrt{a^2 - b})} - \sqrt{\tfrac{1}{2}(a - \sqrt{a^2 - b})}.$$

Note.—In these results, the expressions found for the square root of $a + \sqrt{b}$ and $a - \sqrt{b}$ will be simpler than the original ones only when $\sqrt{a^2 - b}$ has an exact root.

Illustrative Examples.

(1.) Let it be required to find the square root of $9 - 2\sqrt{14}$.

Put $\sqrt{x} - \sqrt{y} = \sqrt{9 - 2\sqrt{14}}$ [1]

And $\sqrt{x} + \sqrt{y} = \sqrt{9 + 2\sqrt{14}}$ [2]

[1] × [2], $x - y = \sqrt{81 - 56} = \sqrt{25} = 5$

By Theorem III., $x + y = 9$

$\therefore\ x = \frac{1}{2}(9 + 5) = 7$

And $y = \frac{1}{2}(9 - 5) = 2$

$\therefore\ \sqrt{9 - 2\sqrt{14}} = \sqrt{7} - \sqrt{2}.$

(2.) Find the square root of $2x + 2\sqrt{x^2 - a^2}$.

Put $\sqrt{m} + \sqrt{n} = \sqrt{2x + 2\sqrt{x^2 - a^2}}$ [1]

And $\sqrt{m} - \sqrt{n} = \sqrt{2x - 2\sqrt{x^2 - a^2}}$ [2]

[1] × [2], $m - n = \sqrt{4x^2 - 4(x^2 - a^2)} = 2a$

By Theorem III., $m + n = 2x$

$\therefore\ m = x + a$ and $n = x - a$

$\therefore\ \sqrt{2x + 2\sqrt{x^2 - a^2}} = \sqrt{x + a} + \sqrt{x - a}.$

EXAMPLES FOR PRACTICE.—LXV.

(1.) Extract the square root of

$$1 - 2x^{\frac{1}{2}} + 5x^{\frac{1}{2}} - 10x^{\frac{3}{4}} + 10x - 12x^{\frac{5}{4}} + 9x^{\frac{3}{2}}.$$

(2.) What is the cube root of

$$8x^3 + 12x^{\frac{5}{3}} - 5x + \tfrac{3}{4}x^{-\frac{1}{3}} - \tfrac{1}{8}x^{-1}?$$

(3.) Find the square roots of $5 + 2\sqrt{6}$, $5 - 2\sqrt{6}$, $13 + 4\sqrt{10}$, and $17 - 4\sqrt{13}$.

(4.) Find the square roots of $4 - \sqrt{15}$, $7 + 3\sqrt{5}$, $\frac{1}{2}(7 + \sqrt{13})$, and $\frac{1}{4}(27 - 10\sqrt{2})$.

(5.) Extract the square roots of $2a - 1 - 2\sqrt{a^2 - a}$ and $x + \sqrt{x^2 - 4y^2}$.

(6.) Simplify $\sqrt{(a+2)^2 - 4(a-2)\sqrt{2a}}$ and

$$\sqrt{\frac{a-b}{2a - \sqrt{4a^2 - b^2}} + \frac{a+b}{2a + \sqrt{4a^2 - b^2}}}.$$

187. The square root of a binomial surd may frequently be found without using the method given above. The following examples will sufficiently show how:—

$$\sqrt{(5 + 2\sqrt{6})} = \sqrt{(3 + 2\sqrt{3 \times 2} + 2)} = \sqrt{3} + \sqrt{2}$$

$$\sqrt{(14 - 6\sqrt{5})} = \sqrt{(14 - 2\sqrt{45})} = \sqrt{(9 - 2\sqrt{9 \times 5} + 5)}$$
$$= 3 - \sqrt{5}$$

$$\sqrt{(4 + \sqrt{7})} = \sqrt{\left(\frac{8 + 2\sqrt{7}}{2}\right)} = \sqrt{\left(\frac{7 + 2\sqrt{7} + 1}{2}\right)} = \frac{\sqrt{7} + 1}{\sqrt{2}}$$

$$\sqrt{(2x - 2\sqrt{x^2 - 1})} = \sqrt{\{(x+1) - 2\sqrt{(x+1)(x-1)} + (x-1)\}}$$
$$= \sqrt{x+1} - \sqrt{x-1}.$$

In each case the expression is first written so that the coefficient of the surd term is 2, and the quantity under

the root is then resolved into factors of which the rational term is the sum. Its root is then obvious.

The student should apply this method to such of the previous examples as will admit of it.

188. Occasionally the square root of a surd expression of the form $a+\sqrt{b}+\sqrt{c}+\sqrt{d}$ may be found by putting

$$\sqrt{a+\sqrt{b}+\sqrt{c}+\sqrt{d}}=\sqrt{x}+\sqrt{y}+\sqrt{z},$$

and squaring each side, which gives

$$a+\sqrt{b}+\sqrt{c}+\sqrt{d}=x+y+z+2\sqrt{xy}+2\sqrt{xz}+2\sqrt{yz}.$$

From this, $a=x+y+z$, $\sqrt{b}=2\sqrt{xy}$, $\sqrt{c}=2\sqrt{xz}$, $\sqrt{d}=2\sqrt{yz}$; and if the values of x, y, and z obtained from the last three equations also satisfy the first, the required square root has been found.

Illustrative Example.

Extract the square root of $9+2\sqrt{6}+4\sqrt{2}+4\sqrt{3}$.

Let $\sqrt{x}+\sqrt{y}+\sqrt{z}=\sqrt{9+2\sqrt{6}+4\sqrt{2}+4\sqrt{3}}$.

Then $x+y+z+2\sqrt{xy}+2\sqrt{xz}+2\sqrt{yz}=9+2\sqrt{6}+4\sqrt{2}+4\sqrt{3}$.

Put $x+y+z=9$, $\sqrt{xy}=\sqrt{6}$, $\sqrt{xz}=2\sqrt{2}$, $\sqrt{yz}=2\sqrt{3}$.

$\sqrt{xy}\times\sqrt{xz}\times\sqrt{yz}=xyz=\sqrt{6}\times 2\sqrt{2}\times 2\sqrt{3}=24$.

$$x=\frac{xyz}{yz}=\frac{24}{12}=2.$$

$$y=\frac{xyz}{xz}=\frac{24}{8}=3.$$

$$z=\frac{xyz}{xy}=\frac{24}{6}=4.$$

Also $x+y+z=2+3+4=9$.

$\therefore$ the required square root is

$$\sqrt{2}+\sqrt{3}+\sqrt{4} \text{ or } \sqrt{2}+\sqrt{3}+2.$$

189. Imaginary Quantities.—Such quantities have been already defined in Art. 138.

Since $-a^2$ cannot be formed by the multiplication of either $+a$ by $+a$ or of $-a$ by $-a$, or of any quantity by itself, it follows that it is not a true square, and that it has no real square root.

The expression $\sqrt{-a^2}$, therefore, which is used to indicate the square root of $-a^2$, is spoken of as an imaginary or impossible quantity.

Since $-a^2$ may be considered as the product of a^2 and -1, $\sqrt{-a^2}$ may be written $\sqrt{a^2(-1)}$ or $a\sqrt{-1}$; so also $\sqrt{-b^2}$ may be written $b\sqrt{-1}$, and $\sqrt{-x}$ as $\sqrt{x}\sqrt{-1}$. It is thus easily seen that all imaginary quantities may be expressed as multiples of $\sqrt{-1}$.

190. The various operations of addition, subtraction, etc., may be performed on imaginary quantities in the same manner as on surds. It is necessary, however, to guard against a very probable error in multiplication.

Let it be required to multiply $\sqrt{-a}$ by $\sqrt{-b}$.

We are apt to write $\sqrt{-a} \times \sqrt{-b} = \sqrt{-a \times -b} = \sqrt{ab}$, whereas it should be $\sqrt{-a} \times \sqrt{-b} = \sqrt{a(-1)} \times \sqrt{b(-1)}$ $= \sqrt{ab(-1)^2} = (-1)\sqrt{ab} = -\sqrt{ab}$.

Similarly we are liable to write that $\sqrt{-a} \times \sqrt{-a}$ $= \sqrt{-a \times -a} = \sqrt{a^2} = \pm a$, instead of $\sqrt{-a} \times \sqrt{-a}$ $= \sqrt{-a \times -a} = \sqrt{(-a)^2} = -a$, there being no dubiety of sign, as a^2 is known to have been derived from $-a$ by $-a$.

General Illustrative Examples.

(1.) Find the sum of $\sqrt{-a^2}$, $2\sqrt{-ab}$, and $\sqrt{-b^2}$.

$\sqrt{-a^2} + 2\sqrt{-ab} + \sqrt{-b^2} = a\sqrt{-1} + 2\sqrt{ab}\sqrt{-1} + b\sqrt{-1}$ $= (a + 2\sqrt{ab} + b)\sqrt{-1} = (\sqrt{a} + \sqrt{b})^2\sqrt{-1}$.

(2.) Multiply $a+b\sqrt{-1}$ by $a-b\sqrt{-1}$.

$$\begin{array}{r} a+b\sqrt{-1} \\ a-b\sqrt{-1} \\ \hline a^2+ab\sqrt{-1} \\ \quad -ab\sqrt{-1}-b^2(-1) \\ \hline a^2 \qquad\quad +b^2 \end{array}$$

191. It appears from this that a^2+b^2 is the product of two factors; and hence we can resolve a binomial whose terms are both positive into factors, one of which is the sum and the other the difference of a real and an imaginary quantity.

This may be considered an extension of Theorem III., Art. 57.

(3.) Resolve 2 into a pair of imaginary factors.

$$2=1+1=1-(-1)=(1+\sqrt{-1})(1-\sqrt{-1}).$$

(4.) Divide $2+3\sqrt{-2}$ by $3-\sqrt{-2}$.

$$\frac{2+3\sqrt{-2}}{3-\sqrt{-2}}=\frac{(2+3\sqrt{-2})}{(3-\sqrt{-2})}\times\frac{(3+\sqrt{-2})}{(3+\sqrt{-2})}=\frac{6+11\sqrt{-2}+3(-2)}{9-(-2)}$$

$$=\frac{11\sqrt{-2}}{11}=\sqrt{-2}.$$

(5.) Raise $\sqrt{-1}$ to the fifth power.

By multiplication—

$$\sqrt{-1}\times\sqrt{-1}=-1, \text{ second power.}$$
$$-1\times\sqrt{-1}=-\sqrt{-1}, \text{ third power.}$$
$$-\sqrt{-1}\times\sqrt{-1}=-(-1)=1, \text{ fourth power.}$$
$$1\times\sqrt{-1}=\sqrt{-1}, \text{ fifth power.}$$

It is plain that higher powers would simply repeat the same series of answers, so that if $x=\sqrt{-1}$, we have only the four forms $x=\sqrt{-1}$, $x^2=-1$, $x^3=-\sqrt{-1}$, and $x^4=1$.

(6.) Extract the square root of $12+2\sqrt{-13}$.

$$\text{Put} \quad \sqrt{x}+\sqrt{y}=\sqrt{12+2\sqrt{-13}}$$
$$\text{And} \quad \sqrt{x}-\sqrt{y}=\sqrt{12-2\sqrt{-13}}$$
$$\therefore \quad x-y=\sqrt{144+52}=\sqrt{196}=14$$
$$\text{Also} \quad x+y=12$$
$$\therefore \quad x=13 \text{ and } y=-1$$
$$\therefore \quad \sqrt{12+2\sqrt{-13}}=\sqrt{13}+\sqrt{-1}.$$

EXAMPLES FOR PRACTICE.—LXVI.

(1.) Find the sum of $\sqrt{-a^2}$, $a-b\sqrt{-1}$, $b-a\sqrt{-1}$, and $\sqrt{-b^2}$; also the difference between $\sqrt{-a^2-2a-1}$ and $\sqrt{-a^2+2a-1}$.

(2.) Multiply $\sqrt{-a^2}$ by $\sqrt{-b^2}$, and $2+\sqrt{-5}$ by $2-\sqrt{-5}$.

(3.) Resolve 5 into two imaginary factors.

(4.) Divide $\frac{1}{2}\sqrt{-3}$ by $\frac{1}{3}\sqrt{-4}$, and $1+\sqrt{-1}$ by $1-\sqrt{-1}$.

(5.) Find the product of $1+\sqrt{-2}+\sqrt{-3}$ by $1+\sqrt{-6}$.

(6.) Cube $-\frac{1}{2}(1-\sqrt{-3})$ and $-\frac{1}{2}(1+\sqrt{-3})$.

(7.) Find the sum of $\frac{1}{2}\cdot\frac{x-y\sqrt{-1}}{x+y\sqrt{-1}}$ and $\frac{1}{2}\cdot\frac{x+y\sqrt{-1}}{x-y\sqrt{-1}}$.

(8.) Expand $(a+b\sqrt{-1})^3$ and $(1-\sqrt{-1})^8$.

(9.) Extract the square root of $5-12\sqrt{-1}$ and of $-5-12\sqrt{-1}$.

(10.) Simplify $\frac{\sqrt{-3}}{\sqrt{3}}+\frac{\sqrt{-3}\sqrt{-5}}{\sqrt{-15}}+\frac{\sqrt{5}}{\sqrt{-5}}$ and

$$\left(\frac{a+2\sqrt{-1}}{\sqrt{-2}}\right)^2+\left(\frac{a-2\sqrt{-1}}{\sqrt{-2}}\right)^2.$$

(11.) Find the continued product of $a+\frac{b}{2}(3+\sqrt{-7})$, $a+\frac{b}{2}(3-\sqrt{-7})$, $a-\frac{b}{2}(3-\sqrt{-7})$, and $a-\frac{b}{2}(3+\sqrt{-7})$.

(12.) Simplify

$$\sqrt{\{\sqrt{2a-1+2a(a-1)\sqrt{-1}}+a^2-a-1+(a+1)\sqrt{-1}\}}.$$

192. Surd Equations. — Examples of these equations have already been given in Chapter X., but they must here be treated somewhat more fully.

Illustrative Examples.

(1.) Given $\sqrt{x+4}+\sqrt{x-3}=1$, to find x.

Squaring, $x+4+2\sqrt{(x+4)(x-3)}+x-3=1$

Collecting, $2x+2\sqrt{x^2+x-12}=0$

Transposing, $\sqrt{x^2+x-12}=-x$

Squaring, $x^2+x-12=x^2$

Transposing, $x=12$.

Remembering what was said in Art. 127 regarding unsatisfactory solutions, we substitute this value of x in the first side of the equation, and obtain—

$$\sqrt{12+4}+\sqrt{12-3}, \text{ or } \sqrt{16}+\sqrt{9}.$$

Extracting the roots, we get—

$$4+3=7,$$

a result which does not correspond with second side, so that $x=12$ does not seem to be a solution of the equation at all.

It must be noticed, however, that $\sqrt{16}$ may be either ± 4 and $\sqrt{9}$ either ± 3, and therefore

$$\sqrt{16}+\sqrt{9}=\pm 4\pm 3=7 \text{ or } -7 \text{ or } 1 \text{ or } -1.$$

It follows from this that by selecting the proper signs on extracting the root, it is possible to have $\sqrt{16}+\sqrt{9}=1$, and consequently $x=12$ is a true solution of the given equation.

If we disregard the double sign, and consider the square root of a quantity as positive only, then $x=12$ is a solution of the equation $\sqrt{x+4}-\sqrt{x-3}=1$, and the one given admits of no arithmetical solution. See Art. 127.

(2.) Find x from the following equation :—

$$\frac{x-\sqrt{x^2-5}}{x+\sqrt{x^2-5}}=2x-1.$$

Multiplying numerator and denominator of first side by $x-\sqrt{x^2-5}$,

$$\frac{x^2-2x\sqrt{x^2-5}+x^2-5}{x^2-(x^2-5)}=2x-1.$$

Simplifying, $x^2-x\sqrt{x^2-5}=5x.$

Dividing out x, and transposing so as to have the surd on a side by itself, $\sqrt{x^2-5}=x-5.$

Squaring, $x^2-5=x^2-10x+25.$

From which $10x=30$ and $x=3.$

From the x divided out, we also get $x=0.$

If these values be substituted in the given equation, it will be found that $x=0$ solves the equation as it stands, while $x=3$ is an arithmetical solution only of the equation

$$\frac{x+\sqrt{x^2-5}}{x-\sqrt{x^2-5}}=2x-1.$$

(3.) Solve the equation

$$\frac{\sqrt{x-3}}{\sqrt{x}}+\sqrt{3}=\frac{\sqrt{x}}{\sqrt{x-3}}$$

Multiplying by $\sqrt{x(x-3)}$, $x-3+\sqrt{3x(x-3)}=x.$

Transposing, $\sqrt{3x(x-3)}=3.$

Squaring, $3x(x-3)=9.$

This yields the quadratic equation $x^2 - 3x = 3$, which being solved in the usual manner gives

$$x = \frac{3 \pm \sqrt{21}}{2}.$$

If these values be substituted in the given equation, it will be found that $x = \dfrac{3 - \sqrt{21}}{2}$ is an arithmetical solution of the equation

$$\frac{\sqrt{x-3}}{\sqrt{x}} - \sqrt{3} = \frac{\sqrt{x}}{\sqrt{x-3}},$$

and not of the above as it stands.

(4.) Find the value of x which satisfies the equation

$$\sqrt[4]{a+x} + \sqrt[4]{a-x} = 2\sqrt[4]{a}.$$

Squaring, $\sqrt{a+x} + 2\sqrt[4]{a^2-x^2} + \sqrt{a-x} = 4\sqrt{a}$.

Transposing, $\sqrt{a+x} + \sqrt{a-x} = 4\sqrt{a} - 2\sqrt[4]{a^2-x^2}$.

Squaring, $2a + 2\sqrt{a^2-x^2} = 16a - 16\sqrt{a}\sqrt[4]{a^2-x^2} + 4\sqrt{a^2-x^2}$.

Transposing, $2\sqrt{a^2-x^2} - 16\sqrt{a}\sqrt[4]{a^2-x^2} + 14a = 0$.

Factoring, $2(\sqrt[4]{a^2-x^2} - 7\sqrt{a})(\sqrt[4]{a^2-x^2} - \sqrt{a}) = 0$.

$$\therefore \sqrt[4]{a^2-x^2} = 7\sqrt{a} \text{ or } \sqrt{a}.$$

Raising to fourth power—

$$a^2 - x^2 = 2401a^2 \text{ or } a^2.$$

$$\therefore x^2 = -2400a^2 \text{ or } 0.$$

$$\text{And } x = 20a\sqrt{-6} \text{ or } 0.$$

Substituting in given equation—

$$\sqrt[4]{a + 20a\sqrt{-6}} + \sqrt[4]{a - 20a\sqrt{-6}} = 2\sqrt[4]{a}.$$

$$\sqrt[4]{a(1 + 20\sqrt{-6})} + \sqrt[4]{a(1 - 20\sqrt{-6})} = 2\sqrt[4]{a}.$$

$$\sqrt[4]{1 + 20\sqrt{-6}} + \sqrt[4]{1 - 20\sqrt{-6}} = 2.$$

$$\sqrt[4]{(1 - \sqrt{-6})^4} + \sqrt[4]{(1 + \sqrt{-6})^4} = 2.$$

$$1 - \sqrt{-6} + 1 + \sqrt{-6} = 2.$$

$$\text{Or } 2 = 2.$$

EXAMPLES FOR PRACTICE.—LXVII.

Find the algebraical solution of the following equations:—

(1.) $\sqrt{x^2-7}=x-1.$

(2.) $\sqrt{x^2+3x+1}+x=2.$

(3.) $\sqrt{16-\sqrt{16x+x^2}}=4-\sqrt{x}.$

(4.) $\dfrac{\sqrt{x}+3}{\sqrt{x}-1}=\dfrac{3\sqrt{x}-1}{3\sqrt{x}-5}.$

(5.) $\dfrac{\sqrt{x+1}+1}{\sqrt{x+1}-1}=\dfrac{x+8}{x}.$

(6.) $\sqrt{1+\sqrt{1+\sqrt{1+2x}}}=\sqrt{1+\sqrt{x}}.$

(7.) $\sqrt{1+x+x^2}+\sqrt{1-x+x^2}=4x.$

(8.) $\sqrt{x^2+8\sqrt{x^2+8\sqrt{x+4}}}=x+4.$

(9.) $\dfrac{\sqrt{x+2}+\sqrt{x+1}}{\sqrt{x+2}-\sqrt{x+1}}+\dfrac{\sqrt{x+2}-\sqrt{x+1}}{\sqrt{x+2}+\sqrt{x+1}}=18.$

(10.) $\sqrt{4x+1}+\sqrt{x+4}=\sqrt{3x+2}+\sqrt{2x+3}.$

(11.) $\sqrt{\{3b^2-ab+b\sqrt{(a^2+b^2+2b\sqrt{ax-ab})}\}}+2b=0.$

(12.) $\sqrt{2x+3}+\sqrt{3x-5}=\sqrt{7x+4}.$

(13.) $\dfrac{1}{\sqrt{1+x}-1}+\dfrac{1}{\sqrt{1-x}+1}=\dfrac{1}{x}.$

(14.) $\dfrac{1+\sqrt{x^2-1}}{1+2a\sqrt{x^2-1}}=\dfrac{1}{1+\sqrt{x^2-1}}.$

(15.) $\sqrt{9-x^2}+2x\sqrt{2}=9\sqrt{1-x^2}.$

(16.) $\sqrt{2x+5}+\sqrt{3x-2}=\sqrt{4x+1}+\sqrt{x+2}.$

(17.) $\dfrac{9x+4}{3\sqrt{x}-2}-3\sqrt{x}=\dfrac{5\sqrt{x}-2}{\sqrt{x}}.$

(18.) $\dfrac{ax+b+\sqrt{a^2x^2+b^2}}{ax+b-\sqrt{a^2x^2+b^2}}=\dfrac{b}{x}$.

(19.) $\left\{\begin{array}{c}\sqrt{\dfrac{x}{y}}+\sqrt{\dfrac{y}{x}}=\dfrac{13}{\sqrt{xy}}+1\\ \sqrt{x}\ +\ \sqrt{y}=5\end{array}\right\}$.

(20.) $\left\{\begin{array}{r}\sqrt{y}-\sqrt{10-x}=\sqrt{y-x}\\ 3\sqrt{10-x}=2\sqrt{y-x}\end{array}\right\}$.

(21.) $\left\{\begin{array}{r}\sqrt{x}-\sqrt{\sqrt{x}+\sqrt{y}}+\sqrt{y}=20\\ 2\sqrt{x}-3\sqrt{y}=5\end{array}\right\}$.

(22.) $\left\{\begin{array}{r}\sqrt{ax}+\sqrt{by}=a+b\\ x+y=2(a+b)\end{array}\right\}$.

(23.) $\sqrt[4]{1+x\sqrt{6}}+\sqrt[4]{1-x\sqrt{6}}=2$.

(24.) $\sqrt[2mn]{x^{m+n}}-\dfrac{1}{2}\cdot\dfrac{a^2-b^2}{a^2+b^2}(\sqrt[m]{x}+\sqrt[n]{x})=0$.

CHAPTER XIV.

RATIO, PROPORTION, AND VARIATION.

193. Ratio.—We may compare one quantity with another of the same kind by ascertaining how much the one is greater than the other, or by ascertaining how many times or parts of a time the one is contained in the other. In this latter method, the relation between the two quantities is called their ratio.

194. If a and b be two quantities of the same kind, the ratio of a to b is obtained by dividing a by b; so that the fraction $\frac{a}{b}$ expresses the ratio of these two quantities. It may also be expressed by writing them with a colon between them, thus, $a : b$. These quantities are spoken of as the terms of the ratio, the former being called the antecedent, the latter the consequent.

When the antecedent is equal to the consequent, we have a ratio of equality.

When the antecedent is greater than the consequent, we have a ratio of greater inequality or of majority.

When the antecedent is less than the consequent, we have a ratio of less inequality or of minority.

Since a ratio may be expressed as a fraction, it follows that if the ratio of a to b be equal to the ratio of c to d, or $a : b = c : d$, that $\frac{a}{b} = \frac{c}{d}$.

195. The following propositions are important:—

PROP. I.—*If two ratios have the same consequent, the first is greater than the second, equal to it, or less, according as the antecedent of the first is greater than that of the second, equal to it, or less.* For if $\frac{a}{c}, \frac{b}{c}$ be the two ratios, treating them as fractions, the first is greater than the second, equal to it, or less, according as a is greater than b, equal to it, or less.

PROP. II.—*A ratio is not altered by having both its terms multiplied or divided by the same quantity.*

By Art. 91, $\frac{a}{b} = \frac{ma}{mb}$, $\therefore a : b = ma : mb$ (Art. 194).

Similarly, $a : b = \frac{a}{n} : \frac{b}{n}$.

From this, and previous proposition, we can compare one ratio with another, so as to ascertain which is the greater.

Illustrative Examples.

(1.) Which is greater, 11 : 18, or 19 : 32?

Change to the fractional form, and reduce to a C.D.

$$\frac{11}{18} = \frac{176}{288} \text{ and } \frac{19}{32} = \frac{171}{288}.$$

Since the fraction $\frac{176}{288} > \frac{171}{288}$, it follows that $\frac{11}{18} > \frac{19}{32}$, and consequently that $11 : 18 > 19 : 32$.

(2.) Compare the ratios $2a+b : a+b$, and $2a-b : a-b$.

$$\frac{2a+b}{a+b} = \frac{2a^2-ab-b^2}{a^2-b^2}, \text{ and } \frac{2a-b}{a-b} = \frac{2a^2+ab-b^2}{a^2-b^2}.$$

Since the second of these fractions exceeds the first by $\frac{2ab}{a^2-b^2}$, $2a-b : a-b$ is greater than $2a+b : a+b$.

PROP. III.—*A ratio of greater inequality is diminished, and one of less inequality is increased, by the addition of the same quantity to each of its terms.*

Let $\frac{a}{b}$ be a given ratio, and $\frac{a+x}{b+x}$ a new one formed by the addition of x to each of the terms of the first.

Reducing to a common denominator, we have—

$$\frac{a}{b} = \frac{ab+ax}{b(b+x)} \text{ and } \frac{a+x}{b+x} = \frac{ab+bx}{b(b+x)}.$$

Now the first of these fractions will be greater or less than the second, according as $ab + ax >$ or $< ab + bx$, that is, as $ax >$ or $< bx$; that is, as $a >$ or $< b$.

If, then, $a > b$, $\frac{a}{b} > \frac{a+x}{b+x}$ { or the ratio is diminished by the addition of x.

And if $a < b$, $\frac{a}{b} < \frac{a+x}{b+x}$ { or the ratio is increased by the addition of x.

PROP. IV.—*A ratio of greater inequality is increased, and one of less inequality is diminished, by the subtraction of the same quantity from each of its terms.*

Let $\frac{a+x}{b+x}$ be a given ratio, then $\frac{a}{b}$ will be the one formed by the subtraction of x from each of the terms of the first.

By reduction, $\frac{a+x}{b+x} = \frac{ab+bx}{b(b+x)}$ and $\frac{a}{b} = \frac{ab+ax}{b(b+x)}$, the second fraction being greater or less than the first,

according as a is greater or less than b, that is, as $a+x$ is greater or less than $b+x$.

If then $a+x>b+x$, $\frac{a+x}{b+x}<\frac{a}{b}$ { or the ratio is increased by the subtraction of x.

And if $a+x<b+x$, $\frac{a+x}{b+x}>\frac{a}{b}$ { or the ratio is diminished by the subtraction of x.

196. When the antecedents of two or more ratios are multiplied together to form a new antecedent, and their consequents to form a new consequent, the resulting ratio is said to be compounded of the others; thus $abc : xyz$ is compounded of the ratios $a:x$, $b:y$, $c:z$, or $\frac{abc}{xyz}=\frac{a}{x}\times\frac{b}{y}\times\frac{c}{z}$.

If $a : b$ be compounded with itself, the result $a^2 : b^2$ is called the duplicate ratio of $a : b$.

And if $a^2 : b^2$ be again compounded with $a : b$, the result $a^3 : b^3$ is called the triplicate ratio of $a : b$.

Similarly, $a^{\frac{1}{2}} : b^{\frac{1}{2}}$ is called the sub-duplicate ratio of $a : b$, and $a^{\frac{1}{3}} : b^{\frac{1}{3}}$ the sub-triplicate ratio of $a : b$.

$a^{\frac{3}{2}} : b^{\frac{3}{2}}$ is sometimes spoken of as the sesquiplicate ratio of $a : b$.

PROP. V.—*If the consequent of one ratio be the antecedent of a second, and the consequent of the second the antecedent of a third, and so on, the ratio compounded of them all is equal to the ratio of the first antecedent to the last consequent.*

Let the given ratios be $\frac{a}{b}, \frac{b}{c}, \frac{c}{d}, \frac{d}{e}, \frac{e}{x}$,

Then $\frac{a}{b}\times\frac{b}{c}\times\frac{c}{d}\times\frac{d}{e}\times\frac{e}{x}=\frac{abcde}{bcdex}=\frac{a}{x}$.

PROP. VI.—*A ratio is increased by being compounded with a ratio of greater inequality, and diminished by being compounded with one of less inequality.*

Let $\frac{a}{b}$ be compounded with $\frac{m}{n}$, then $\frac{a}{b} \times \frac{m}{n} = \frac{am}{bn}$.

But $\frac{a}{b} = \frac{an}{bn}$. (Art. 91.) Now $\frac{am}{bn}$ will be greater or less than $\frac{an}{bn}$ according as m is greater or less than n.

If, then, $m > n$, $\frac{am}{bn} > \frac{a}{b}$ { or the ratio is increased by being compounded with $\frac{m}{n}$.

And if $m < n$, $\frac{am}{bn} < \frac{a}{b}$ { or the ratio is diminished by being compounded with $\frac{m}{n}$.

197. Approximate Ratios.—When one quantity differs very slightly from another, the ratio between their powers or their roots may be expressed approximately.

Let $a \pm b$ and a be two quantities differing very slightly, and consequently such that b is very small as compared with a; then

$$\begin{aligned}(a \pm b)^2 : a^2 &= a^2 \pm 2ab + b^2 : a^2 \\ &= a \pm 2b + \frac{b^2}{a} : a \\ &= a \pm 2b : a,\end{aligned}$$

since $\frac{b^2}{a}$ must be a very small fraction, and may be neglected.

Similarly, $$\begin{aligned}(a \pm b)^3 : a^3 &= a^3 \pm 3a^2b + 3ab^2 \pm b^3 : a^3 \\ &= a \pm 3b + \frac{3b^2}{a} \pm \frac{b^3}{a^2} : a \\ &= a \pm 3b : a,\end{aligned}$$

and generally $(a \pm b)^n : a^n = a \pm nb : a$.

By assuming that this holds true also when n is a fraction, or by actually extracting the roots of the given quantities, we have

$$\sqrt{a \pm b} : \sqrt{a} = a \pm \tfrac{1}{2}b : a$$
$$\sqrt[3]{a \pm b} : \sqrt[3]{a} = a \pm \tfrac{1}{3}b : a$$
and generally $$\sqrt[n]{a \pm b} : \sqrt[n]{a} = a \pm \frac{1}{n}b : a.$$

198. Incommensurable Quantities.—It frequently happens that the relation between two quantities cannot be exactly expressed by any number, either whole or fractional. When this occurs, the quantities are said to be incommensurable. Thus, if the side of a square be represented by 1, its diagonal will be represented by $\sqrt{2}$ and the ratio between them by $\frac{1}{\sqrt{2}}$, to which no exact numerical value can be given, as 2 has no square root. But its root can be obtained approximately to any degree of accuracy, and therefore a fraction can be found which shall differ from $\frac{1}{\sqrt{2}}$ by a quantity less than any that can be assigned. This fraction may be considered equal to $\frac{1}{\sqrt{2}}$, and will represent the given ratio. It thus appears that the definition of a ratio, although strictly applicable only to quantities that have a common unit of measurement, may be extended so as to include those that are incommensurable.

General Illustrative Examples.

(1.) Compare the ratios 53 : 81, and 41 : 63.

$\frac{53}{81} > < \frac{41}{63}$, but $\frac{53}{81} = \frac{371}{567}$ and $\frac{41}{63} = \frac{369}{567}$.

(2.) Show that $a:b > ax:bx+y$, but $< ax:bx-y$.

$\frac{a}{b} > < \frac{ax}{bx+y}$, but $\frac{a}{b} = \frac{abx+ay}{b(bx+y)}$ and $\frac{ax}{bx+y} = \frac{abx}{b(bx+y)}$.

$$\therefore \frac{a}{b} > \frac{ax}{bx+y}.$$

$\frac{a}{b} > < \frac{ax}{bx-y}$, but $\frac{a}{b} = \frac{abx-ay}{b(bx-y)}$ and $\frac{ax}{bx-y} = \frac{abx}{b(bx-y)}$.

$$\therefore \frac{a}{b} < \frac{ax}{bx-y}.$$

(3.) Find the simple ratio of $\frac{35}{27} : \frac{25}{18}$.

$$\frac{35}{27} \div \frac{25}{18} = \frac{35}{27} \times \frac{18}{25} = \frac{7}{3} \times \frac{2}{5} = \frac{14}{15}.$$

(4.) Compound the ratios 72 : 91 and 78 : 99, and reduce the result to its simplest form.

$$\frac{72}{91} \times \frac{78}{99} = \frac{8}{7} \times \frac{6}{11} = \frac{48}{77}.$$

(5.) Find the approximate ratio between 1000^3 and 999^3.

Since $a^3 : (a-b)^3 = a : a - 3b$ nearly,

$1000^3 : 999^3 = 1000 : 1000 - 3$ nearly,

$= 1000 : 997$ nearly.

The exact ratio is 1000 : 997·002999.

(6.) Solve the equations—

$$ax + by + cz = 0 \qquad [1]$$

$$\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 0 \qquad [2]$$

$$a^2b^2c^2xyz + b^3c^3x + a^3c^3y + a^3b^3z = 0 \qquad [3]$$

Obtain from [1] and [2] the ratios that x, y, and z bear to one another.

$[1]\times ab,\quad a^2bx+ab^2y+abcz=0\quad[4]$
$[2]\times abc^2,\quad bc^2x+ac^2y+abcz=0\quad[5]$
$[4]-[5],\quad b(a^2-c^2)x+a(b^2-c^2)y=0$

$$\therefore\ \frac{x}{a(b^2-c^2)}=\frac{y}{b(c^2-a^2)}.$$

Similarly $$\frac{y}{b(c^2-a^2)}=\frac{z}{c(a^2-b^2)}.$$

$$\therefore\ \frac{x}{a(b^2-c^2)}=\frac{y}{b(c^2-a^2)}=\frac{z}{c(a^2-b^2)};$$

l if each of these ratios $= m$, we have—

$$x=a(b^2-c^2)m,\ y=b(c^2-a^2)m,\ z=c(a^2-b^2)m.$$

bstitute in [3]—

$$x^3b^3c^3(b^2-c^2)(c^2-a^2)(a^2-b^2)m^3+ab^3c^3(b^2-c^2)m$$
$$+ba^3c^3(c^2-a^2)m+ca^3b^3(a^2-b^2)m=0.$$

vide out $abcm$—

$$x^2b^2c^2(b^2-c^2)(c^2-a^2)(a^2-b^2)m^2+b^2c^2(b^2-c^2)+a^2c^2(c^2-a^2)$$
$$+a^2b^2(a^2-b^2)=0,$$

, $a^2b^2c^2(b^2-c^2)(c^2-a^2)(a^2-b^2)m^2=(b^2-c^2)(c^2-a^2)(a^2-b^2)$

$$\therefore\ a^2b^2c^2m^2=1,\text{ and } m=\pm\frac{1}{abc}.$$

From this,

$$=\pm\left(\frac{b}{c}-\frac{c}{b}\right),\ y=\pm\left(\frac{c}{a}-\frac{a}{c}\right),\text{ and } z=\pm\left(\frac{a}{b}-\frac{b}{a}\right).$$

EXAMPLES FOR PRACTICE.—LXVIII.

(1.) Which is the greater ratio, 17 : 36 or 23 : 48 ?

(2.) Find the ratio compounded of 15 : 16 and 24 : 25.

(3.) What is the simple ratio of $\frac{28}{33}:\frac{35}{44}$?

(4.) Write the triplicate ratio of 3 to 5, and the subplicate ratio of $6\frac{1}{4}:9$.

(5.) Compound the ratios $a-b:a+b$, $a^2-b^2:a^2+b^2$, d $a^4-b^4:(a-b)^4$.

(6.) Find approximately the ratio of $\left(\frac{1}{100}\right)^{10}$ to $\left(\frac{1}{99}\right)^{10}$, and of 200^5 to 201^5.

(7.) Two numbers are in the ratio of 5 to 6, and when 3 is taken from each the remainders are in the ratio of 4 to 5. What are the numbers?

(8.) If $\frac{a}{b} > \frac{c}{d} > \frac{e}{f}$, show that $\sqrt{\frac{a^2+c^2+e^2}{b^2+d^2+f^2}}$ is less than $\frac{a}{b}$ and greater than $\frac{e}{f}$.

(9.) What must be added to each term of the ratio $a^3 : b^3$ to make it equal to the ratio $a : b$?

(10.) Solve the equations, $6x+y-3z=0$, $7x-8y+2z=0$, $x^4+y^4+z^4=11552$.

(11.) One piece of metal consists of 2 parts copper to 3 parts tin, another consists of 5 parts copper to 7 parts tin: in what ratio must the pieces be combined so as to form a mixture of 9 parts copper to 13 parts tin?

(12.) Solve the equations—

$$abx+acy+bcz=0$$
$$(a+b)x+(a+c)y+(b+c)z=0$$
$$a^4b^4x^2+a^4c^4y^2+b^4c^4z^2=2a^2b^2c^2(a^3b^2x+a^2c^3y+b^3c^2z).$$

199. Proportion.—When the ratio between one pair of quantities is equal to the ratio between another pair, the four quantities are said to be proportional to one another.

Thus, when $\frac{a}{b} = \frac{c}{d}$, a, b, c, and d are proportionals; the first being the same multiple or part of the second that the third is of the fourth. Their relation to one another is usually expressed by writing them as follows:—

$$a : b = c : d, \text{ or } a : b :: c : d,$$

this being read a is to b as c is to d.

The first and last terms of this statement are called the extremes, the second and third the means.

200. When the second term is the same as the third, it is said to be a mean proportional between the other two; thus in $a : b :: b : c$, b is a mean proportional between a and c, and c is spoken of as a third proportional to a and b.

201. When in a number of equal ratios the consequent of one is always the antecedent of the next, the terms of these ratios are said to be in continued proportion; thus—

If $a : b = b : c$, $b : c = c : d$, and $c : d = d : e$, then a, b, c, d, and e are in continued proportion.

202. The terms of a proportion need not be all of the same kind of magnitude, but each consequent must be of the same kind as its antecedent. Thus, 3 cwt. : 4 tons :: 6 shillings : £8 would be a proper proportion, while 3 cwt. : 6 shillings :: 4 tons : £8 would not.

203. Propositions in Proportion.

PROP. I.—*When four quantities are proportionals, the product of the extremes is equal to the product of the means.*

Let a, b, c, d be proportionals, then $\frac{a}{b} = \frac{c}{d}$, and multiplying both sides by bd, we have $ad = bc$.

From this we can find any term of a proportion when the other three are known, for $a = \frac{bc}{d}$, $b = \frac{ad}{c}$ $c = \frac{ad}{b}$, and $d = \frac{bc}{a}$, this last being the foundation of the ordinary rule for working simple proportion in Arithmetic.

PROP. II.—*When the product of two quantities is equal*

to that of other two, the four quantities are proportional; the terms of either product forming the means, and those of the other the extremes.

Let $ad = bc$, then dividing both sides by bd, we have $\frac{a}{b} = \frac{c}{d}$ or $a : b :: c : d$.

PROP. III.—*When four quantities are proportional, they are also proportional when taken alternately; or, the first is to the third as the second is to the fourth.*

If $a : b :: c : d$, then $ad = bc$ (Prop. I.) Dividing by dc, $\frac{a}{c} = \frac{b}{d}$, and $\therefore a : c :: b : d$.

This will only hold true when the quantities are all of one kind.

PROP. IV.—*When four quantities are proportional, they are also proportional when taken inversely; or, the second is to the first as the fourth is to the third.*

If $a : b :: c : d$, then $ad = bc$, or $bc = ad$. Dividing by ac, $\frac{b}{a} = \frac{d}{c}$, and $\therefore b : a :: d : c$.

PROP. V.—*When three quantities are in continued proportion, the product of the first and third is equal to the square of the second, and conversely.*

If $a : b :: b : c$, then $ac = b^2$ (Prop. I.) Conversely, if $ac = b^2$, then $a : b :: b : c$ (Prop. II.)

PROP. VI.—*When three quantities are in continued proportion, the first has to the third the duplicate ratio of the first to the second.*

If $a : b :: b : c$, then $b^2 = ac$ (Prop. V.) Multiplying both sides by a, $ab^2 = a^2c$, and $\therefore a : c :: a^2 : b^2$ (Prop. II.)

Or, $a : c ::$ dup. $a : b$.

Similarly, we can show that if a, b, c, d are in continued proportion, then $a : d :: a^3 : b^3$, and generally if there are n terms, $a : z :: a^{n-1} : b^{n-1}$.

PROP. VII.—*When four quantities are proportional, if the first and second be multiplied or divided by any quantity, as likewise the third and fourth, the results are proportional.*

If $a : b :: c : d$, then $\frac{a}{b} = \frac{c}{d}$, but $\frac{a}{b} = \frac{ma}{mb}$ and $\frac{c}{d} = \frac{nc}{nd}$,

$\therefore \frac{ma}{mb} = \frac{nc}{nd}$, and $ma : mb :: nc : nd$, m and n being whole or fractional.

PROP. VIII.—*Also if the first and third be multiplied or divided by any quantity, and likewise the second and fourth, the results are proportional.*

As before, $\frac{a}{b} = \frac{c}{d}$

Multiplying by $\frac{m}{n}$, $\frac{ma}{nb} = \frac{mc}{nd}$, and $\therefore ma : nb :: mc : nd$.

This will evidently hold true when either m or n is unity.

PROP. IX.—*When four quantities are proportional, and likewise other four, if the corresponding terms of the two sets be multiplied together, the results are proportional.*

Given $a : b :: c : d$ and $e : f :: g : h$ then $\frac{a}{b} = \frac{c}{d}$ and $\frac{e}{f} = \frac{g}{h}$.

Multiplying, $\frac{ae}{bf} = \frac{cg}{dh}$, and $\therefore ae : bf :: cg : dh$.

This is called compounding, and may be extended to any number of proportions.

PROP. X.—*When four quantities are proportional, the like powers and roots are also proportional.*

As before, $\frac{a}{b}=\frac{c}{d}$, and squaring, $\frac{a^2}{b^2}=\frac{c^2}{d^2}$, or $a^2:b^2::c^2:d^2$.

Similarly, $\frac{a^{\frac{1}{2}}}{b^{\frac{1}{2}}}=\frac{c^{\frac{1}{2}}}{d^{\frac{1}{2}}}$, or $a^{\frac{1}{2}}:b^{\frac{1}{2}}::c^{\frac{1}{2}}:d^{\frac{1}{2}}$,

and generally $a^n:b^n::c^n:d^n$, whether n be whole or fractional.

The theorems proved in Art. 109 may be held to establish the following propositions, a, b, c, d being four quantities in proportion :—

PROP. XI.—*The sum of the first and second is to the second as the sum of the third and fourth is to the fourth.*

By Theorem I., Art. 109, $\frac{a+b}{b}=\frac{c+d}{d}$,

and $\therefore\ a+b:b::c+d:d$.

This is called *Composition*, or *Componendo*.

PROP. XII.—*The difference between the first and second is to the second as the difference between the third and fourth is to the fourth.*

Theorem II., $\frac{a-b}{b}=\frac{c-d}{d}$, $\therefore\ a-b:b::c-d:d$.

This is called *Division*, or *Dividendo*.

PROP. XIII.—*The sum of the first and second is to their difference as the sum of the third and fourth is to their difference.*

Theorem III., $\frac{a+b}{a-b}=\frac{c+d}{c-d}$, $\therefore\ a+b:a-b::c+d:c-d$.

This is called *Composition* and *Division*, or sometimes *Mixing*.

PROP. XIV.—*When several quantities are in continued proportion, any one of the antecedents is to its consequent as the sum of all the antecedents is to the sum of all the consequents.*

Theorem IX., $\frac{a}{b} = \frac{ma+nc+pe}{mb+nd+pf}$, and if $m=n=p=1$, then $\frac{a}{b} = \frac{a+c+e}{b+d+f}$, $\therefore a:b::a+c+e:b+d+f$.

The proposition may easily be extended to any number of proportionals.

There are many other propositions in proportion, which may be proved in a manner similar to the above, but they must be left as exercises to the student.

General Illustrative Examples.

(1.) Prove that a, b, c, d are proportionals when $(a+b+c+d)(a-b-c+d) = (a-b+c-d)(a+b-c-d)$.

Expressing as fractions, $\frac{a+b+c+d}{a+b-c-d} = \frac{a-b+c-d}{a-b-c+d}$.

By Art. 109, $\frac{a+b}{c+d} = \frac{a-b}{c-d}$ (Theorem III.)

Also $\frac{a}{c} = \frac{b}{d}$ (Theorem VII.), or $\frac{a}{b} = \frac{c}{d}$ (Theorem V.)

$\therefore a:b::c:d$.

(2.) If a, b, c, d are proportional, show that $a^2bd+b^2c+bc = ab^2c+abd+ad$.

Put $a^2bd+b^2c+bc > = < ab^2c+abd+ad$.

Since $a:b::c:d$, $bc=ad$.

Subtracting equals, $a^2bd+b^2c > = < ab^2c+abd$.

Dividing by b, $a^2d+bc > = < abc+ad$.

Subtracting equals, $a^2d > = < abc$.

Dividing by a, $ad=bc$.

$\therefore a^2bd+b^2c+bc = ab^2c+abd+ad$.

(3.) The times in which two boatmen row the same distance up a stream are as a to b, and their times of rowing equal distances down are as c to d. Find the ratio of their rates of rowing in still water.

Let x = rate of first per hour in miles.
y = rate of second per hour in miles.
z = rate of stream per hour in miles.

If m be the number of miles they row up, and n the number they row down, then—

$$\frac{m}{x-z} : \frac{m}{y-z} :: a : b \qquad [1]$$

And
$$\frac{n}{x+z} : \frac{n}{y+z} :: c : d \qquad [2]$$

From [1], $y-z : x-z :: a : b$, and $ax = by + (a-b)z$.
From [2], $y+z : x+z :: c : d$, and $cx = dy + (d-c)z$.

From these equations we obtain—

$$(ad - 2ac + bc)x = (2bd - ad - bc)y$$

And $\therefore \quad \frac{x}{y} = \frac{2bd - ad - bc}{ad - 2ac + bc}$.

EXAMPLES FOR PRACTICE.—LXIX.

(1.) Prove that ratios which are equal to the same ratio are equal to one another. (This is called *Equality of ratios* or *Ex Æqualæ.*)

(2.) Find a fourth proportional to $\sqrt{6}$, $\sqrt{3}$, and $\sqrt{2}$, and a third proportional to $\sqrt{3}$ and $\sqrt{6}$.

(3.) Find a mean proportional to $\sqrt{13}+2$ and $\sqrt{13}-2$.

(4.) If x be a mean proportional to 7 and 1183, what is its value?

(5.) Given $a:b::c:d$, to show that $a^2d^2 + abc = a^2d + b^2c^2$.

(6.) Find x, when $a+x : b+x :: a^{\frac{1}{2}} : b^{\frac{1}{2}}$.

(7.) In a school there has recently been an increase of a per cent. in the number of boys, of b per cent. in the number of girls, and of c per cent. on the whole. Find the relative number of boys to girls at present in school.

(8.) Given a, b, c in continued proportion, to show that

$$(a+b+c)(a-b+c)=a^2+b^2+c^2; \quad [1]$$

And that $a+c>2b$. [2]

(9.) If four quantities be proportional, prove that the sum of the greatest and the least is greater than the sum of the other two.

(10.) The distances which two boatmen can row in equal times in still water are as 10 to 9. If the times they take to row equal distances in going up a river are as 4 to 5, what is the ratio of the times in which they would row equal distances going down?

(11.) Find the ratio of x to y when

$$\frac{\sqrt[n]{x}-a\sqrt[n]{y}}{\sqrt[n]{x}+a\sqrt[n]{y}}=\frac{\sqrt[2n]{x}-a\sqrt[2n]{x-\frac{3}{2}y}}{\sqrt[2n]{x}+a\sqrt[2n]{x-\frac{3}{2}y}}.$$

(12.) If the six quantities a, b, c, d, e, f are such that $a:b::c:d$, and $e:b::f:d$, show that $a+e:c+f::b:d$, and also that $a^m+e^m:c^m+f^m::e^m:f^m$.

204. Variation.—Questions in ratio and proportion are sometimes presented in a manner somewhat different from that given in the preceding pages.

When $A:B::C:D$, $\frac{A}{B}=\frac{C}{D}$, and as the ratio expressed is a constant quantity, we may put it equal to m.

This gives $\frac{A}{B}=m$, and $\therefore A=mB$.

If now, in this equation, any alteration takes place in the value of B, a proportional alteration will take place in that of A. If B be doubled, tripled, or quadrupled in value, so will A; and if B be halved, quartered, etc., so will A.

A depends for its value on B, and is said to *vary as* B.

The relation between them is generally expressed thus, $A \propto B$, the character $\propto$ being read *varies as.*

205. When one quantity increases as another increases, or diminishes as the other diminishes, the first is said to *vary directly as* the second.

$A \propto B$ implies that A varies directly as B.

When one quantity increases as another diminishes, or diminishes as the other increases, the first is said to *vary inversely as* the second.

$A \propto \frac{1}{B}$ implies that A varies inversely as B, for if we put $A = m \cdot \frac{1}{B}$, then as B grows larger, A grows less, and as B grows less, A grows larger.

When a quantity varies as the product of two or more quantities, it is said to *vary as* these others *jointly.*

$A \propto BC$ indicates that A varies directly as the product of B and C.

When a quantity varies as the quotient of other two, it is said to *vary directly as* the dividend, and *inversely as* the divisor.

$A \propto \frac{B}{C}$ indicates that A varies directly as B, and inversely as C.

206. Propositions in Variation.

PROP. I.—If $A \propto B$, and $B \propto C$, then $A \propto C$.

Put $A = mB$, and $B = nC$.

Then $A = mnC$, or $A \propto C$, mn being constant.

Similarly, if $A \propto B$, and $B \propto \frac{1}{C}$, then $A \propto \frac{1}{C}$.

PROP. II.—If $A \propto \frac{1}{B}$, and $B \propto \frac{1}{C}$, then $A \propto C$.

$$A = \frac{m}{B}, \text{ and } B = \frac{n}{C}.$$

$$\therefore A = \frac{\frac{m}{n}}{C} = \frac{m}{n}C, \text{ or } A \propto C, \frac{m}{n} \text{ being constant.}$$

PROP. III.—If $A \propto C$, and $B \propto C$, then $A \pm B \propto C$.

$$A = mC, \; B = nC.$$
$$\therefore A \pm B = (m \pm n)C,$$
or, $A \pm B \propto C$, $m \pm n$ being constant.

PROP. IV.—If $A \propto C$, and $B \propto C$, then $AB \propto C^2$.

$$A = mC, \; B = nC, \; \therefore AB = mnC^2,$$
or, $AB \propto C^2$, mn being constant.

Corollary, $C \propto \sqrt{AB}$, for $C = \frac{1}{\sqrt{mn}}\sqrt{AB}$.

PROP. V.—If $A \propto BC$, then $B \propto \frac{A}{C}$, and $C \propto \frac{A}{B}$.

$$A = mBC, \; \therefore B = \frac{1}{m} \cdot \frac{A}{C}, \text{ and } C = \frac{1}{m} \cdot \frac{A}{B},$$
and consequently $B \propto \frac{A}{C}$, and $C \propto \frac{A}{B}$.

PROP. VI.—If $A \propto B$ and $C \propto D$, then $AC \propto BD$ and $\frac{A}{C} \propto \frac{B}{D}$.

$A = mB$, $C = nD$, $\therefore AC = mnBD$ or $AC \propto BD$.

Also $\frac{A}{C} = \frac{m}{n}\frac{B}{D}$, or $\frac{A}{C} \propto \frac{B}{D}$.

PROP. VII.—If $A \propto B$ when C is constant, and $A \propto C$ when B is constant, then $A \propto BC$.

When C is constant, mC will also be constant, and we may put $A = mC \cdot B$.

Similarly, when B is constant, we may put $A = nB \cdot C$.

Multiplying, $A^2 = mnB^2C^2$.

$\therefore\ A = \sqrt{mn}BC$, or $A \propto BC$.

PROP. VIII.—If $A \propto B$ when C and D are constant, and $A \propto C$ when B and D are constant, and $A \propto D$ when B and C are constant, then $A \propto BCD$.

For C and D constant, we may put $A = mCD \cdot B$.

For B and D constant, we may put $A = nBD \cdot C$.

For B and C constant, we may put $A = pBC \cdot D$.

$\therefore\ A^3 = mnpB^3C^3D^3$, or $A \propto BCD$.

A similar result may be proved for any number of quantities.

General Illustrative Examples.

(1.) If $ax - by = cx - dy$, prove that $x \propto y$.

Transposing, $(a - c)x = (b - d)y$

$$\therefore\ x = \frac{b-d}{a-c}y, \text{ and as } \frac{b-d}{a-c} \text{ is constant, } x \propto y.$$

(2.) Given that $y^2 \propto x^2 + b^2$, and that when $x = (a^2 - b^2)^{\frac{1}{2}}$, $y = \frac{a^2}{b}$, to find the equation between x and y.

Put $y^2 = m(x^2 + b^2)$, then $m = \frac{y^2}{x^2 + b^2}$.

Substituting for x and y, their given values, we have $m = \frac{a^4}{b^2} \div (a^2 - b^2 + b^2) = \frac{a^2}{b^2}$.

From this, $y^2 = \frac{a^2}{b^2}(x^2 + b^2)$, and $\therefore\ b^2y^2 = a^2x^2 + a^2b^2$, the required equation.

If now any value be given to x, the corresponding value of y may be found.

Suppose $x = b$, then $b^2y^2 = a^2b^2 + a^2b^2$, and $y = \pm a\sqrt{2}$.

(3.) Given that y is the sum of three quantities, of which the first is constant, while the second varies as x and the third as x^2. If, when x equals 1, 2, 3 successively, the values of y are 9, 15, 23, what is the equation between x and y?

Let $y = m + nx + px^2$ represent the required equation.

Substituting successively the given values of x, we have

$$\begin{aligned} m + n + p &= 9 \\ m + 2n + 4p &= 15 \\ m + 3n + 9p &= 23, \end{aligned}$$

three equations from which we can find m, n, and p.

Solving them in the usual way, we get—

$$m = 5,\ n = 3, \text{ and } p = 1.$$

The required equation is $\therefore\ y = 5 + 3x + x^2$.

(4.) If $(a+b+c)(a+b-c)(a+c-b)(b+c-a)$ varies as a^2b^2, then $a^2+b^2-c^2$ is either 0, or it varies as ab.

Put $(a+b+c)(a+b-c)(a-b+c)(-a+b+c) = ma^2b^2$.

$$\{(a+b)+c\}\{(a+b)-c\}\{(a-b)+c\}\{(a-b)-c\} = -ma^2b^2.$$

$$\{(a+b)^2-c^2\}\{(a-b)^2-c^2\} = -ma^2b^2.$$

$$(a^2-b^2)^2 - 2(a^2+b^2)c^2 + c^4 = -ma^2b^2.$$

Add $4a^2b^2$ to each side—

$$(a^2+b^2)^2 - 2(a^2+b^2)c^2 + c^4 = 4a^2b^2 - ma^2b^2 = a^2b^2(4-m).$$

Then $a^2+b^2-c^2 = \pm ab\sqrt{4-m}$.

When $m = 4$, $a^2+b^2-c^2 = 0$.

For any other value of m, $a^2+b^2-c^2 \propto ab$.

EXAMPLES FOR PRACTICE.—LXX.

(1.) If $4x - 3y = 3x + 2y$, show that $x \propto y$.

(2.) Given that $y \propto x$, and that when $x = 3$, $y = 12$, to find the value of y when $x = 5$.

(3.) It is known that y is the sum of two quantities, the first of which varies directly as x^2 and the second in-

versely as x. What is the equation between x and y, if, when $x=1$, $y=7$, and when $x=2$, $y=14$?

(4.) If $y \propto x^2+9$, and if when $x=4$, $y=100$, what value will y have when $x=\frac{1}{2}$?

(5.) Given that $x \propto y$, to prove that $x^2+xy+y^2 \propto y^2$.

(6.) If $a+b \propto a-b$, show that $a^2+b^2 \propto ab$.

(7.) If several sums of money, each lent for a different length of time, all gain the same interest (simple), show that the times must vary inversely as the interest on each sum for one year.

(8.) Show that x^2-y^2 is constant if $\frac{x}{y} \propto x+y$, and $\frac{y}{x} \propto x-y$.

(9.) If several triangles have the same area, their bases vary inversely as their altitudes. Prove this.

(10.) In what time will 3 men and 11 boys perform a piece of work which can be done by 9 men and 13 boys in 3 days, or by 5 men and 9 boys in 5 days?

(11.) The speed of an engine is 20 miles an hour when it draws two carriages, and 16 miles when it draws eight, its speed being reduced by a quantity which varies as the square root of the number of carriages attached. Find its rate by itself, and the greatest number of carriages it can draw.

(12.) Given that $s \propto t^2$ when f is constant, and $s \propto f$ when t is constant, also that $2s=f$ when $t=1$, to find the equation formed by f, s, t.

CHAPTER XV.

ARITHMETICAL, GEOMETRICAL, AND HARMONICAL PROGRESSION.

207. Arithmetical Progression. — When several quantities increase or diminish according to a fixed law, they are said to form a series. When this law is such that the difference between every two consecutive terms is the same, the series is called an Arithmetical Progression.

Thus, 3, 8, 13, 18, 23, and 11, 8, 5, 2, are arithmetical progressions, the first *increasing* regularly by 5, the second *diminishing* regularly by 3.

The difference between the successive terms is called the common difference.

Terms standing between two other terms of an arithmetical progression are called the *arithmetical means* between those terms. One is spoken of as the *arithmetical mean.*

208. PROP. I.—*To find any term of an arithmetical progression.*

If a represent the first term of an arithmetical progression, and d the common difference, then a, $a+d$,

$a+2d$, $a+3d$, etc., will form an Increasing Arithmetical Progression, and a, $a-d$, $a-2d$, $a-3d$, etc., a Decreasing Arithmetical Progression.

Notice that one term differs from another only in the coefficient of d, and that each coefficient is one less than the number of the term in which it occurs. Thus in the third term the coefficient of d is 2, and in the fourth term 3; in the tenth term it will be 9, and in the nth term $n-1$. The nth term will therefore be $a \pm (n-1)d$, the plus or minus sign being taken according as the series is increasing or decreasing. $a+(n-1)d$ is called the general term, and may be considered to include $a-(n-1)d$, so that if l be put for the nth term, we shall have the equation $l=a+(n-1)d$.

209. PROP. II.—*To find the sum of an arithmetical progression.*

If l be the last term, the second last will be $l-d$, the third last $l-2d$, and so on.

If s represent the sum of the series, we shall have—

$$s=a+(a+d)+(a+2d)+\ldots+(l-2d)+(l-d)+l,$$

and also

$$s=l+(l-d)+(l-2d)+\ldots+(a+2d)+(a+d)+a$$

by writing the series backwards.

Adding the two lines together, term by term, we obtain—

$$2s=(a+l)+(a+l)+\ldots+(a+l)+(a+l)$$

as many times as there are terms in the series; and if there be n terms, then—

$$2s=n(a+l), \text{ and } s=\frac{n}{2}(a+l).$$

If now we substitute in this expression the value of l found in Prop. I., we shall obtain—

$$s=\frac{n}{2}\{a+a+(n-1)d\}=\frac{n}{2}\{2a+(n-1)d\}.$$

Collecting the above results, we have—

$$l=a+(n-1)d. \qquad \text{I.}$$

$$s=\frac{n}{2}(a+l). \qquad \text{II.}$$

$$\text{and } s=\frac{n}{2}\{2a+(n-1)d\}. \qquad \text{III.}$$

From these formulæ any two of the quantities a, d, l, n, and s can be found, if the other three are given.

Illustrative Examples.

(1.) Find the seventh term of the series 1, 4, 7, 10, etc.

Subtracting any of the terms from the succeeding one, we find the common difference to be 3, and as the first term is 1, we have

$$a=1,\ d=3,\ n=7,$$

and $\therefore$ by substitution in I.,

$$l=a+(n-1)d=1+(7-1)\times 3=1+6\times 3=1+18=19.$$

(2.) Find the sum of eight terms of the series 13, 9, 5, etc.

Here $9-13=-4=d$, $a=13$, $n=8$.

Substitute in III.,

$$\therefore\ s=\frac{n}{2}\{2a+(n-1)d\}$$

$$=\frac{8}{2}\{26-7\times 4\}=4\times -2=-8.$$

(3.) The first term of an arithmetical progression is 5, the last 23, and the number of terms is 13; what is the common difference?

Substituting in I., $23 = 5 + (13 - 1)d$.

Transposing, $12d = 18$, and $\therefore\ d = 1\frac{1}{2}$.

(4.) The sum of the series 10, 17, 24, etc., is 1122; how many terms are there?

Here $d = 7$.

Substituting in III.,

$$1122 = \frac{n}{2}\{20 + (n-1) \times 7\}$$

$$= \frac{n}{2}(20 - 7 + 7n),$$

$$\text{Or, } 7n^2 + 13n = 2244.$$

Solving, $n = 17$ or $-18\frac{6}{7}$, and, as the second value is inapplicable, the number of terms is 17.

Occasionally, both values of n will afford a solution; thus, if $a = 10$, $d = -2$, and $s = 24$, we shall have the equation $n^2 - 11n + 24 = 0$, from which we get $n = 3$ or 8.

Writing the series $10 + 8 + 6 + 4 + 2 + 0 - 2 - 4$, we see that $s = 24$, either for $n = 3$ or $n = 8$.

Again, if $a = 9$, $d = 3$, and $s = 54$, we obtain the equation $n^2 + 5n - 36 = 0$, from which we derive $n = 4$ or -9.

In this case we have $9 + 12 + 15 + 18 = 54$, and also $-6 - 3 + 0 + 3 + 6 + 9 + 12 + 15 + 18 = 54$, the nine terms being counted backwards, from the last term of the positive series.

(5.) The first, fifth, and last terms of an arithmetical progression are 3, 19, and 35 respectively. Find the common difference, the number of terms, and the sum of the series.

From I. we have $d = \dfrac{l - a}{n - 1}$, and considering the fifth term to be the last, then

$$d = \frac{19 - 3}{5 - 1} = \frac{16}{4} = 4.$$

Also from I., $n = \dfrac{l - a + d}{d}$,

$$\text{and } n = \frac{35 - 3 + 4}{4} = \frac{36}{4} = 9.$$

Substituting in II., $s = \frac{9}{2}(3 + 35) = 9 \times 19 = 171.$

(6.) Insert four arithmetical means between $\frac{3}{16}$ and $\frac{4}{3}$; and find the sum of 16 terms of the series.

In the first part of the question, counting the two end terms and the four means, we have $n = 6$; and substituting in I.,

$$\frac{4}{3} = \frac{3}{16} + (6 - 1)d, \text{ or } 64 = 9 + 240d,$$

$$\text{and } \therefore\ d = \frac{55}{240} = \frac{11}{48}.$$

Adding this difference to the first, second, and other terms successively, we obtain the four means required—namely,

$$\frac{20}{48}, \frac{31}{48}, \frac{42}{48}, \text{ and } \frac{53}{48}.$$

To find the sum, substitute in III.,

$$s = \frac{16}{2}\left\{\frac{6}{16} + (16 - 1) \times \frac{11}{48}\right\} = \frac{1}{2}(6 + 55) = 30\tfrac{1}{2}.$$

210. Arithmetical Progression is occasionally called Equidifferent Progression.

EXAMPLES FOR PRACTICE.—LXXI.

(1.) Find the thirteenth term of the series 2, 5, 8, 11, etc.

(2.) What is the nth term of a series which decreases by 4, the first term being 4?

(3.) The first term of an arithmetical progression is $\frac{7}{8}$, and the third $\frac{2}{3}$; what is the tenth?

(4.) Find the sum of the numbers 1, 2, 3, 4, etc., to n and also to 1000 terms.

(5.) Find the sum of the series 37, 30, 23, etc., to eleven and also to fifteen terms.

(6.) If the first term of an arithmetical progression be 1, and the common difference $\frac{2}{3}$, what is the sum of n and of fifty-one terms?

(7.) Find the nth term, and the sum of n terms of the odd numbers 1, 3, 5, 7, etc.

(8.) What is the common difference when the fifth term is 37 and the twelfth 100?

(9.) Find the common difference of the decreasing arithmetical progression in which the nth term is a and the mth term b; a being $> b$.

(10.) Insert three arithmetical means between 47 and 91.

(11.) Insert five arithmetical means between $2\frac{1}{3}$ and $-\frac{2}{3}$.

(12.) Show that the sum of the first and last terms of an arithmetical progression is the same as the sum of any two terms equally distant from them.

(13.) The first and last terms of an arithmetical progression are 3 and 37, and the sum is 420. How many terms are there?

(14.) If the sum of the series 50, 46, 42, etc., be 50, what number of terms does it contain?

(15.) Prove that in an arithmetical progression of an uneven number of terms, the first and last terms together are double the middle one.

(16.) The middle term of an arithmetical progression of thirty-one terms is 35, and the common difference is 2. Find the series.

(17.) The first term of an arithmetical progression is 30, its middle term 1, and the common difference $-2\frac{5}{12}$. Find the number of terms and their sum.

(18.) The sum of the first three terms of an arithmetical progression is 30, and of the second three 66. What is the sum of the following six?

(19.) Find three numbers in arithmetical progression such that their sum shall be 93, and the product of first and last 861.

(20.) If a person lay past one penny on the first day of the year, twopence on the second, threepence on the third, and so on to the end of the year, what will his savings amount to?

(21.) A train starts from a certain station at noon, and proceeds on its journey at the uniform rate of 35 miles an hour. Two hours later another sets out after it, at the rate of 30 miles an hour, but at the end of every hour it increases its rate by 5 miles. When will the second overtake the first?

(22.) If s_1, s_2, s_3, s_4 be respectively the sums of n terms of the arithmetical progressions 1, 2, 3 etc., 1, 3, 5 etc., 1, 4, 7 etc., and 1, 5, 9 etc., show that s_1, s_2, s_3, s_4 are themselves in arithmetical progression, and find their common difference.

(23.) If a falling body passes through $16\frac{1}{12}$ feet in the first second, $48\frac{3}{12}$ feet in the next, $80\frac{5}{12}$ feet in the third, and so on, what time will it take to fall a distance of 2316 feet? Compare the space through which it falls in the last second with that through which it falls in the first.

(24.) A regiment of 1200 men, employed in the construction of a fortification, was divided into a number of equal working parties. On the first day one party was sent out; on the second, two; on the third, three; and so on till all were out; and then they continued

working together till the work was finished. Had the number of parties been reduced by placing a man more in each, they would have completed the work one day sooner, while if the whole had been employed from the beginning, they could have done it in $35\frac{1}{2}$ days. How many parties were sent out, and how long was the first engaged?

211. Double Arithmetical Progression.

Illustrative Examples.

(1.) Find the sum of $1-3+5-7+9-$ etc. to twenty terms.

This may be considered to be composed of the difference between two arithmetical progressions, and its sum may be found by taking the sum of the one from that of the other. Thus the sum of twenty terms of the above series will be—

$$1+5+9+\text{etc. to ten terms} - (3+7+11+\text{etc. to ten terms}).$$

$$= \frac{10}{2}\{2+9\times 4\} - \frac{10}{2}\{6+9\times 4\}$$

$$= 5\times 38 - 5\times 42 = -20.$$

It may also be found by performing the subtraction indicated on the terms as they stand; the above then becomes $-2-2-2-$ etc. to ten terms, that is, -20.

When the number of terms is odd, the sum may be found in the same way by setting the first term aside, and taking the difference between second and third, fourth and fifth, etc.

Suppose the sum of the above to be required to twenty-one terms, then we have $1+(2+2+2+\text{etc. to ten terms}) = 21$.

(2.) Find the sum of n terms of the series $4-7+10-13+16-$ etc.

When n is even, the number of differences is $\frac{n}{2}$.

$$\therefore\ s = -3-3-3-\text{etc. to } \frac{n}{2} \text{ terms} = -\frac{3}{2}n.$$

When n is odd, the first term being left out, the number of differences will be $\frac{n-1}{2}$.

$$\therefore\ s = 4+(3+3+3+\text{etc. to } \frac{n-1}{2} \text{ terms}) = 4+\frac{3}{2}(n-1)$$
$$= \frac{3n+5}{2}.$$

Half the sum of the two answers is

$$\frac{1}{2}\left(\frac{3n+5}{2}-\frac{3n}{2}\right) = \frac{5}{4}.$$

Half their difference is

$$\frac{1}{2}\left(\frac{3n+5}{2}+\frac{3n}{2}\right) = \frac{6n+5}{4}.$$

Connecting these terms together as under, we have an expression which represents the sum of the series, whether n be odd or even—

$$s = \frac{1}{4}\{5-(6n+5)(-1)^n\}.$$

(3.) Find the sum of the series $1\cdot3+4\cdot5+7\cdot7+10\cdot9+$ etc. to n terms.

The factors of the terms here form the two series—

$$1+4+7+10+\ldots+(3n-2),$$
and $$3+5+7+9+\ldots+(2n+1).$$

So that the general term (Art. 208) of the given series is $(3n-2)(2n+1)$.

Assume that

$$s = 1\cdot3+4\cdot5+\ldots+(3n-5)(2n-1)+(3n-2)(2n+1)$$
$$= An+Bn^2+Cn^3+\text{etc.} \qquad \text{I.}$$

Change n into $n+1$, and rewrite the series—

$$1\cdot3+4\cdot5+\ldots+(3n-2)(2n+1)+(3n+1)(2n+3)$$
$$=A(n+1)+B(n+1)^2+C(n+1)^3. \quad \text{II.}$$

Taking I. from II. we have—

$$(3n+1)(2n+3)=A+B(2n+1)+C(3n^2+3n+1),$$

or $6n^2+11n+3=3Cn^2+(2B+3C)n+A+B+C.$

Equate the coefficients of like powers of n.

This gives $6=3C$, $11=2B+3C$, and $3=A+B+C$.

From which $C=2$, $B=\frac{5}{2}$ and $A=-\frac{3}{2}$.

Substituting these values in I., we have—

$$s=-\frac{3}{2}n+\frac{5}{2}n^2+2n^3=\frac{n}{2}(4n^2+5n-3).$$

The same method may be applied to find the sum of series such as $2\cdot3\cdot4+4\cdot5\cdot6+6\cdot7\cdot8+$ etc., $1^2+2^2+3^2+$ etc., $1^3+3^3+5^3+$ etc.

(4.) Find the sum of $2\cdot5-3\cdot9+4\cdot13-5\cdot17+6\cdot21-7\cdot25+$ etc. to $2r$ and also to $2r+1$ terms.

Writing the series so as to admit of the subtraction indicated, we have—

$$s=10-27+52-85+126-175+\text{etc. to } 2r \text{ terms.}$$
$$=\quad -17\quad -33\quad -49+\text{etc. to } r \text{ terms.}$$

Here $a=-17$ and $d=-16$.

$$\therefore\ s=\frac{r}{2}\{-34+(r-1)\times-16\}=-r(8r+9).$$

This may also be found by the method of last example, for by it the sum of r of the positive terms is $\frac{1}{3}(16r^3+15r^2-r)$, and r of the negative terms is $\frac{1}{3}(16r^3+39r^2+26r)$, and

$$\therefore\ s=\frac{1}{3}(16r^3+15r^2-r-16r^3-39r^2-26r)$$
$$=\frac{1}{3}(-24r^2-27r)=-r(8r+9).$$

Again—

$s = 10 - 27 + 52 - 85 + 126 - 175 + 232 - \text{etc. to } (2r+1) \text{ terms},$

$= 10 \quad + (25 \quad + 41 \quad + 57 + \text{etc. to } r \text{ terms}),$

$= 10 + \frac{r}{2}\{50 + (r-1)16\} = 10 + r(8r+17).$

If $n =$ either $2r$ or $2r+1$, we have from the first answer $s = -\frac{n}{2}(4n+9)$, and from the second $s = 10 + \frac{n-1}{2}\{4(n-1)+17\}$

$= \frac{1}{2}(4n^2 + 9n + 7).$

Connecting half the sum of these with half their difference, as in Example 2, we have—

$$s = \frac{1}{4}\{7 - (8n^2 + 18n + 7)(-1)^n\},$$

a general expression for n terms of the above series, whether n be odd or even.

EXAMPLES FOR PRACTICE.—LXXII.

(1.) Find the general term of the series $3 - 8 + 13 - 18 + 23 - \text{etc.}$

(2.) Find the sum of the series $1 - 5 + 9 - 13 + 17 - 21 + \text{etc.}$ [I.] to ten terms; [II.] to thirteen terms; [III.] to n terms, whether n be odd or even.

(3.) What is the nth term of the series $2 \cdot 3 - 4 \cdot 5 + 6 \cdot 7 - 8 \cdot 9 + 10 \cdot 11 - \text{etc.}$?

(4.) Find the sum of the series $1 - 3 + 6 - 10 + 15 - 21 +$ etc. to n terms.

(5.) How many terms of the series $1 - 7 + 13 - 19 + 25 -$ etc. amount to -192? How many to 175?

(6.) Find the nth term and the sum of n terms of the series $1 + 3 + 6 + 10 + 15 + \text{etc.}$

(7.) Sum to n terms the series whose general term is $3n^2 - 1$.

(8.) What is the sum of $2r$ terms of the series $1\cdot3-2\cdot5+3\cdot7-4\cdot9+5\cdot11-6\cdot13+$ etc.; also of $2r+1$ terms, and of n terms when n is either odd or even?

(9.) Find the sum of the series $1^2+3^2+5^2+7^2+$ etc. to n terms.

(10.) Sum n terms of the series $1\cdot2\cdot5+3\cdot4\cdot7+5\cdot6\cdot9+7\cdot8\cdot11+$ etc.

(11.) Show that $1^3+2^3+3^3+\ldots+n^3$ is equal to $(1+2+3+\ldots+n)^2$.

(12.) If $\{a+(n-1)d\}\{a_1+(n-1)d_1\}$ be the general term of a series, prove that its sum may be expressed by $\frac{n}{6}\{6aa_1+3(n-1)(ad_1+a_1d)+(n-1)(2n-1)dd_1\}$.

212. Geometrical Progression.—When in a series the ratio between every two consecutive terms is the same, the series is called a Geometrical Progression. Thus, 3, 6, 12, 24, and 9, 3, 1, $\frac{1}{3}$, $\frac{1}{9}$ are geometrical progressions, in the first of which the *common ratio* is 2, and in the second $\frac{1}{3}$.

Terms standing between any two terms of a geometrical progression are called the *geometrical means* between those terms. One is spoken of as the *geometrical mean*.

213. PROP. I.—*To find any term of a geometrical progression.*

Let a represent the first term of a geometrical progression, and r the *common ratio*, then the terms of the series will be a, ar, ar^2, ar^3, etc., which only differ from one another in the exponent of r; each exponent being *one* less than the number of the term in which it stands. Thus, in the third term it is 2, in the fifth term it will be 4, and in the nth term $n-1$.

The nth term is therefore ar^{n-1}, and if l be the nth term, $l=ar^{n-1}$. [I.]

214. PROP. II.—*To find the sum of a geometrical progression.*

If s represent the sum of the series, and n the number of terms, we shall have—

$$s = a + ar + ar^2 + ar^3 + \ldots + ar^{n-3} + ar^{n-2} + ar^{n-1} \text{ [A]},$$
$$\text{and } rs = ar + ar^2 + ar^3 + ar^4 + \ldots + ar^{n-2} + ar^{n-1} + ar^n \text{ [B]},$$

by multiplying the whole by r.

With the exception of the first term of the one series and the last of the other, the second sides of the two expressions are identical, and by subtracting [A] from [B] we have—

$$(r-1)s = a(r^n - 1)$$
$$\text{and } \therefore\ s = \frac{a(r^n-1)}{r-1} \quad \text{[II.]}$$

As $l = ar^{n-1}$, this may be written—

$$s = \frac{rl - a}{r-1}.$$

215. PROP. III.—*To find the sum to infinity of a decreasing geometrical series.*

In the expression [II.], when r is less than unity, numerator and denominator will both be negative, and it will be preferable to write the value of s thus :—

$$s = \frac{a(1-r^n)}{1-r}.$$

Also, since r is less than unity, its powers will be successively less and less, until, by taking n large enough, r^n will be less than any quantity that can be named. It may therefore be neglected, and, putting Σ (sigma) to represent the altered value, we may write

$$\Sigma = a\left(\frac{1}{1-r}\right) \text{ or } \frac{a}{1-r}. \quad \text{[III.]}$$

Now taking n large implies a great number of terms in the series, and if we suppose this number to be infinitely great, then when r is a proper fraction $\Sigma = \frac{a}{1-r}$ is the sum of the series to infinity.

It must be borne in mind, however, that $\frac{a}{1-r}$ is greater than the true sum by an infinitely minute quantity, which, although it may be lessened by taking more terms of the series, can never be got rid of. $\frac{a}{1-r}$ is, therefore, the limit to which the series continually tends, although practically it is the sum of the series, and is usually spoken of as such.

Illustrative Examples.

(1.) What is [1] the twelfth term of the series 2, 6, 18, etc., and [2] the twentieth of 256, 128, 64, 32, etc.?

The common ratio r is found by dividing the second term by the first, or the third by the second, etc. This gives for the first series $r = 3$, for the second $r = \frac{1}{2}$.

Substituting in I., we have—

$$[1], \quad l = 2 \times 3^{11} = 2 \times 177147 = 354294.$$

$$[2], \quad l = 256 \times \frac{1}{2^{19}} = \frac{256}{524288} = \frac{1}{2048}.$$

(2.) Find the sum of $1 - 5 + 25 - 125 + \text{etc.}$ to nine terms.

Here $a = 1$, $r = -5$, and $n = 9$.

Substituting in II.—

$$s = \frac{(-5)^9 - 1}{-5 - 1} = \frac{-1953126}{-6} = 325521.$$

(3.) Find the sum of ten terms of the series

$$\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \text{etc.}$$

Here $a=\frac{1}{2}$, $r=\frac{1}{2}$, and $n=10$.

$$\therefore\ s=\frac{\frac{1}{2}\left(1-\frac{1}{2^{10}}\right)}{1-\frac{1}{2}}=\frac{2^{10}-1}{2^{10}}=\frac{1023}{1024}.$$

Observe here that the sum is a fraction very little less than 1; and if more and more terms be taken, it will approach more and more nearly to 1, but will never reach it or exceed it, even if an infinite number of terms be taken, for the numerator must always be less than the denominator. We say, therefore, that 1 is the *limit* of the above series.

We arrive at the same result by substitution in III.; thus,

$$\Sigma=\frac{\frac{1}{2}}{1-\frac{1}{2}}=\frac{1}{1}=1.$$

Similarly we find the *limit* or *sum to infinity* of the series $\frac{2}{3}+\frac{4}{9}+\frac{8}{27}+$ etc. to be 2; for substituting in III. we have—

$$\Sigma=\frac{2}{3}\left(\frac{1}{1-\frac{2}{3}}\right)=\frac{2}{3}\left(\frac{3}{3-2}\right)=\frac{2}{3}\cdot\frac{3}{1}=2.$$

(4.) Insert four geometrical means between $\frac{1}{25}$ and 125.

We may consider these the first and last terms of a geometrical progression of six terms; this gives

$$a=\frac{1}{25},\ n=6,\text{ and } l=125,\text{ to find } r.$$

By I., $$l=ar^{n-1}=\frac{1}{25}r^5=125.$$

$$\therefore\ r^5=25\times125=5^5,\text{ and } r=5.$$

The series is $\therefore\ \frac{1}{25},\ \frac{1}{5},\ 1,\ 5,\ 25,\ 125.$

EXAMPLES FOR PRACTICE.—LXXIII.

(1.) Find the seventh term of the series 2, 10, 50, etc., and the ninth of the series $\frac{3}{4}, \frac{1}{2}, \frac{1}{3}$, etc.

(2.) By how much does the fifth term of the series $\frac{27}{32}, -\frac{3}{4}, \frac{2}{3}$, etc., differ from the sixth?

(3.) Write the next five terms of the geometrical progression $\frac{1}{4} - \frac{\sqrt{-1}}{2\sqrt{2}}$.

(4.) Find the sum of $1+4+16+$ etc., and of $1-3+9-$ etc., each to seven terms.

(5.) Find the sum of $1+\frac{1}{5}+\frac{1}{25}+$ etc., and of $1-\frac{1}{4}+\frac{1}{16}-$ etc., each to ten terms.

(6.) What is the sum of n terms of each of the series $3-6+12-$ etc. and $\frac{3}{4}+1+\frac{4}{3}+$ etc.?

(7.) What is the sum of n terms of the series $1+\frac{2}{x}+\frac{4}{x^2}+$ etc., and of $\frac{x}{y}-\frac{x^2}{2y^2}+\frac{x^3}{4y^3}-$ etc.?

(8.) Insert three geometric means between 12 and 972, and five between $\frac{27}{8}$ and $\frac{8}{27}$.

(9.) Find the geometrical mean, or *mean proportional*, to 228 and 513, also to $\sqrt{1+a^2}-\sqrt{1-a^2}$ and $\sqrt{1+a^2}+\sqrt{1-a^2}$.

(10.) Find the *limit*, or *sum to infinity*, of the follow-

ing series :—[I.] $1+\frac{1}{2}+\frac{1}{4}+$etc. ; [II.] $\frac{1}{3}+\frac{1}{9}+\frac{1}{27}+$etc. ; [III.] $\frac{1}{2}+\frac{1}{3}+\frac{2}{9}+$etc. ; [IV.] $\frac{2}{3}+\frac{1}{2}+\frac{3}{8}+$etc.

(11.) Sum to infinity [I.] $\frac{2}{3}-\frac{1}{3}+\frac{1}{6}-$etc. ; [II.] $\frac{1}{10}-\frac{1}{10^2}+\frac{1}{10^3}-$etc. ; [III.] $3\frac{1}{8}-1\frac{1}{4}+\frac{1}{2}-$etc. ; [IV.] $-1+\frac{7}{8}-\frac{7^2}{8^2}+$etc.

(12.) Find the value of ·5555. . . , and of ·23454545. . . , each to infinity.

(13.) The population of a town was 65700 in 1871, and 67890 in 1881 ; what should it become in 1891, if the same rate of increase be maintained ?

(14.) The sum of two numbers is 180, and their mean proportional is 54. What are they ?

(15.) Show that any term of a geometrical progression continued to infinity is greater than all that come after it if r is $< \frac{1}{2}$, and equal to them if $r=\frac{1}{2}$.

(16.) From a vessel containing at first 64 gallons of wine, a certain quantity of liquor is drawn off daily, and is replaced with water. At the end of six days it is found that 11 gallons $3\frac{1}{8}$ pints of wine still remain in the mixture. What quantity is drawn per day ? and how much wine is in the last drawing ?

(17.) The first term of a geometrical progression, continued to infinity, is 1, and every term is equal to the sum of the succeeding ones. Find the series. Also find the series when every term is five times the sum of all succeeding ones.

(18.) If a be the first term and l the last, r the common ratio, and s the sum of n terms of a geometrical progression, show that the following relations are true :—

$$[\text{I.}]\ r=\frac{s-a}{s-l};\ [\text{II.}]\ s=\frac{\sqrt[n-1]{l^n}-\sqrt[n-1]{a^n}}{\sqrt[n-1]{l}-\sqrt[n-1]{a}};$$
$$\text{and } [\text{III.}]\ l(s-l)^{n-1}-a(s-a)^{n-1}=0.$$

216. Mixed Progressions.

Illustrative Examples.

(1.) Find the sum of the series
$aA+(a+d)Ar+(a+2d)Ar^2+\ldots+\{a+(n-1)d\}Ar^{n-1}$.

This is formed by the products of the corresponding terms of the arithmetical and geometrical series—

$$a+(a+d)+(a+2d)+\ldots+\{a+(n-1)d\}$$
and
$$A+Ar+Ar^2+\ldots+Ar^{n-1}.$$

Put $s=aA+(a+d)Ar+(a+2d)Ar^2+\ldots+\{a+(n-1)d\}Ar^{n-1}$

then $rs=aAr+(a+d)Ar^2+\ldots+\{a+(n-2)d\}Ar^{n-1}+\{a+(n-1)d\}Ar^n$

and $(1-r)s=aA+dAr+dAr^2+\ldots+dAr^{n-1}-\{a+(n-1)d\}Ar^n$

$$=aA+dAr\left(\frac{1-r^{n-1}}{1-r}\right)-\{a+(n-1)d\}Ar^n.$$

$$\therefore\ s=\frac{A}{1-r}\left[a+dr\left(\frac{1-r^{n-1}}{1-r}\right)-\{a+(n-1)d\}r^n\right].$$

If r be a fraction and n infinite, then $\Sigma=\frac{A}{1+r}\left(a+\frac{dr}{1-r}\right)$.

(2.) Sum the series whose general term is $a\left(\frac{1-b^n}{1-b}\right)Ar^{n-1}$.

Setting aside the constant multiplier aA, we have

$s=1+(1+b)r+(1+b+b^2)r^2+\ldots+(1+b+\ldots+b^{n-1})r^{n-1}$

then $brs=br+(b+b^2)r^2+\ldots+(b+\ldots+b^{n-1})r^{n-1}+(b+b^2+\ldots+b^n)r^n$

and $(1-br)s=1+r+r^2+\ldots+r^{n-1}-(b+b^2+\ldots+b^n)r^n$.

$$= \frac{1-r^n}{1-r} - \frac{b(1-b^n)r^n}{1-b}$$

$$s = \frac{1}{1-br}\left\{\frac{1-r^n}{1-r} - \frac{b(1-b^n)r^n}{1-b}\right\}.$$

$$\therefore\ S = \frac{aA}{1-br}\left\{\frac{1-r^n}{1-r} - \frac{b(1-b^n)r^n}{1-b}\right\};$$

$$\text{and } \Sigma = \frac{aA}{(1-br)(1-r)}.$$

EXAMPLES FOR PRACTICE.—LXXIV.

(1.) Find the sum of n terms of the series $1\cdot1+2\cdot2+3\cdot4+4\cdot8+\ldots.+n\times2^{n-1}$.

(2.) Find the sum of n terms of $3\cdot3+7\cdot9+11\cdot27+15\cdot81+$ etc.

(3.) Find the sum of n terms of $4\cdot1-7\cdot4+10\cdot16-\ldots+(3n+1)(-4)^{n-1}$.

(4.) Sum to n terms, and also to infinity, $\frac{1}{2}+\frac{3}{4}+\frac{5}{8}+\ldots+\frac{2n-1}{2^n}$.

(5.) Sum $4+44+444+$ etc. to n terms.

(6.) If $A_1=a$, $A_2=a+am$, $A_n=a+am+\ldots am^{n-1}$, what is the sum of the series $A_1+A_2r^2$, $+A_3r^4+\ldots+A_nr^{2n-2}$ to n terms, and also to infinity?

217. Geometrical Progression is sometimes called Equirational Progression.

218. Harmonical Progression.—A number of quantities are said to be in Harmonical Progression when, of any three consecutive terms, the first is to the third as the

difference between the first and second is to the difference between the second and third,—the differences being taken in the same order. Thus, 3, 4, 6, 12 are in Harmonical Progression,

$$\text{For } 3 : 6 :: 4-3 : 6-4$$
$$\text{And } 4 : 12 :: 6-4 : 12-6.$$

Terms intermediate between any two terms of a harmonical progression are said to be harmonical means to those two terms.

219. PROP. I.—*Quantities in Harmonical Progression have their reciprocals in Arithmetical Progression.*

Let a, b, c, d be in harmonical progression,

$$\text{Then } a : c :: a-b : b-c \qquad \text{[Def.]}$$
$$\text{And } ab-ac = ac-bc. \qquad \text{(Art. 203).}$$

Dividing by abc, $\frac{1}{c} - \frac{1}{b} = \frac{1}{b} - \frac{1}{a}$.

Similarly, since $b : d :: b-c : c-d$,

$$\text{We have } bc-bd = bd-cd$$
$$\text{And} \qquad \frac{1}{d} - \frac{1}{c} = \frac{1}{c} - \frac{1}{b}.$$
$$\therefore \frac{1}{d} - \frac{1}{c} = \frac{1}{c} - \frac{1}{b} = \frac{1}{b} - \frac{1}{a}.$$

That is, $\frac{1}{a}$, $\frac{1}{b}$, $\frac{1}{c}$, $\frac{1}{d}$ are in arithmetical progression, for the difference between every consecutive two is the same.

220. PROP. II.—*To insert a number of harmonical means between any two quantities.*

Let a and b be the given quantities, and m the required number of means. Since (Prop. I.) the reciprocals of a harmonical progression are equidifferent, we may write those of the required series thus :—

$$\frac{1}{a},\ \frac{1}{a}+d,\ \frac{1}{a}+2d \ldots . \frac{1}{a}+md,\ \frac{1}{a}+(m+1)d,$$

and the harmonical series will be

$$a,\ \frac{a}{1+ad},\ \frac{a}{1+2ad} \ldots . \frac{a}{1+mad},\ \frac{a}{1+(m+1)ad}.$$

As the last term is b, we have

$$\frac{a}{1+(m+1)ad}=b, \text{ from which } d=\frac{a-b}{(m+1)ab}.$$

The harmonical means are therefore

$$\frac{(m+1)ab}{mb+a},\ \frac{(m+1)ab}{mb+2a-b}, \ldots . \frac{(m+1)ab}{ma+b}.$$

221. PROP. III.—*To find any term of a harmonical series when two consecutive terms are known.*

Let a and b be the first and second terms of a harmonical series. Find the nth.

Here $\frac{1}{a}, \frac{1}{b}$ are the first two terms of an arithmetical series (Prop. I.) whose common difference is $\frac{1}{b}-\frac{1}{a}$, and whose nth term is $\frac{1}{a}+(n-1)\left(\frac{1}{b}-\frac{1}{a}\right)$ or $\frac{(n-1)a-(n-2)b}{ab}$.

The nth term of the harmonical series is therefore $\frac{ab}{(n-1)a-(n-2)b}$.

222. *Note.*—The sum of a harmonical series cannot be

reduced to a general form, but must be obtained in each case by actual addition.

223. It may be useful to bear in mind that any three quantities a, b, c are in arithmetical, geometrical, or harmonical progression according as

$$\frac{a-b}{b-c}=\frac{a}{a}, \text{ or } =\frac{a}{b} \text{ or } =\frac{a}{c}.$$

Also that $\frac{1}{2}(a+c)$ is the arithmetical mean,

$\sqrt{ac}$ is the geometrical mean,

and $\frac{2ac}{a+c}$ is the harmonical mean,

between the two quantities a and c.

Illustrative Examples.

(1.) Find five harmonical means between $\frac{1}{5}$ and $\frac{1}{23}$.

The series here must consist of seven terms whose reciprocals are 5, $5+d$, $5+2d$, etc., the seventh being $5+6d=23$.

This gives $d=3$, and therefore the reciprocals are 5, 8, 11, 14, 17, 20, 23, and the harmonical series—

$$\frac{1}{5}, \frac{1}{8}, \frac{1}{11}, \frac{1}{14}, \frac{1}{17}, \frac{1}{20}, \frac{1}{23}.$$

(2.) The arithmetical mean between two quantities is a, and the harmonical mean is b; what are the quantities?

Let x = the first, y = the second.

By Art. 223 the arithmetical mean between x and y is $\frac{x+y}{2}$; and the harmonical mean is $\frac{2xy}{x+y}$.

$$\therefore \frac{x+y}{2}=a \text{ [1], and } \frac{2xy}{x+y}=b \text{ [2].}$$

Multiplying together, we have $xy = ab$ [3]

Squaring [1], $x^2 + 2xy + y^2 = 4a^2$

[3] × 4, $4xy = 4ab$

Subtracting, $x^2 - 2xy + y^2 = 4a(a-b)$

$\therefore\ x - y = \pm 2\sqrt{a(a-b)}$

But $x + y = 2a.$

$\therefore\ x = a \pm \sqrt{a(a-b)}$, and $y = a \mp \sqrt{a(a-b)}$.

EXAMPLES FOR PRACTICE.—LXXV.

(1.) Find the next three terms of the harmonical series $\frac{1}{10}, \frac{2}{17}, \frac{3}{21}$.

(2.) Find the fifth, seventh, and tenth terms of the harmonical progression of which the first two terms are $\frac{1}{4}$ and $\frac{3}{10}$.

(3.) Insert three harmonical means between 1 and 101.

(4.) Place four harmonical means between 2 and -3.

(5.) The arithmetical mean between two numbers is 15, and the harmonical mean is $9\frac{3}{5}$; what are the numbers?

(6.) The harmonical mean between two numbers is $7\frac{1}{5}$, and the geometrical mean is 12; find the numbers.

(7.) The sum of three fractions in harmonical progression is $\frac{11}{12}$, and the sum of their squares is $\frac{49}{144}$. What are the fractions?

(8.) If a, b, c be in arithmetical progression, and b, c, d in harmonical progression, prove that $a : b :: c : d$.

(9.) Show that the geometrical mean between two quantities is a *mean proportional* between the arithmetical and the harmonical means, and that of these last the arithmetical mean is the greater.

(10.) The arithmetical mean between two numbers exceeds the geometrical mean between them by 6, and the geometrical mean exceeds the harmonical by $4\frac{4}{5}$. Find the numbers.

(11.) Three quantities are in harmonical progression. If half the middle term be taken from each, the remainders are in geometrical progression. Prove this.

(12.) If a, b, c be respectively the pth, qth, and rth terms of a harmonical series, show that

$$\frac{p-q}{c} + \frac{r-p}{b} + \frac{q-r}{a} = 0.$$

CHAPTER XVI.

INDETERMINATE EQUATIONS.

224. It has been shown (Art. 112) that a single equation containing two unknown quantities yields an unlimited number of solutions, since the substitution of any value for the one letter gives a corresponding value for the other.

We may, however, restrict the number of these solutions (sometimes to a very few) by agreeing to admit only those values that fulfil certain conditions, such as that they shall be positive, or integral, or both.

Illustrative Examples.

(1.) Find integral values of x and y from the indeterminate equation,

$$5x + 8y = 31.$$

Transfer to second side the unknown quantity having the largest coefficient, and divide by the coefficient of the other, thus

$$5x = 31 - 8y,$$

and

$$x = \frac{31 - 8y}{5} = 6 - y + \frac{1 - 3y}{5}.$$

Multiply $\frac{1-3y}{5}$ by such a number as will render the coefficient of y in it one more or less than some multiple of the denominator.

Suitable multipliers may generally be found by inspection.

In this example any one of the following may be used, 2, 3, 7, 8, 12, 13, etc.

Taking the least, we have

$$2 \times \frac{1-3y}{5} = \frac{2-6y}{5} = -y + \frac{2-y}{5}.$$

Let $\frac{2-y}{5} = m$, then $2 - y = 5m$, from which $y = 2 - 5m$,

and $$x = \frac{31 - 8(2-5m)}{5} = 3 + 8m.$$

By putting successive integral values for m, we shall find any number of integral values of x and y.

When $m = 2$, $y = -8$, and $x = 19$.
When $m = 1$, $y = -3$, and $x = 11$.
When $m = 0$, $y = 2$, and $x = 3$.
When $m = -1$, $y = 7$, and $x = -5$.
When $m = -2$, $y = 12$, and $x = -13$.
. . . . . .

Observe that the values of m, y, and x form three arithmetical series, whose common differences are respectively 1, 5, and 8.

In the above series there is only one instance in which the values of x and y are positive at the same time, so that if positive as well as integral values had been required, there would have been only one solution of the equation.

(2.) Find the positive integral values of x and y that satisfy the equation

$$7x + 4y = 153.$$

$$y = \frac{153 - 7x}{4} = \frac{152 - 8x + 1 + x}{4}$$

$$= 38 - 2x + \frac{1 + x}{4}.$$

Put $\frac{1+x}{4} = m$, then $x = 4m - 1$,

and $y = \frac{153 - 7(4m-1)}{4} = 40 - 7m.$

Here, in order that x may be positive, m must be greater than $\frac{1}{4}$; and in order that y may be positive, m must be less than $5\frac{5}{7}$.

As the values must also be integral, m cannot be less than *one* or greater than *five*, so that the only possible values of m are 1, 2, 3, 4, and 5, from which we have—

When $m = 1$, $x = 3$, and $y = 33$.
When $m = 2$, $x = 7$, and $y = 26$.
When $m = 3$, $x = 11$, and $y = 19$.
When $m = 4$, $x = 15$, and $y = 12$.
When $m = 5$, $x = 19$, and $y = 5$.

(3.) In how many ways can the sum of £24 be paid in guineas and crowns?

Let x = number of guineas, y = number of crowns,

then $21x + 5y = 480.$

$$y = \frac{480 - 21x}{5} = 96 - 4x - \frac{x}{5}.$$

Put $\frac{x}{5} = m$, then $x = 5m$,

and $y = 96 - 20m - m = 96 - 21m.$

Here x will be positive for any positive value of m, but y will not be positive for any positive integral value greater than 4.

The only possible values of m are therefore 1, 2, 3, 4, so that the sum can be paid in four ways.

(4.) Given $13x - 17y = 8$, to find positive integral values of x and y.

Here $$x = \frac{17y+8}{13} = y + \frac{4y+8}{13}$$

$$3 \times \frac{4y+8}{13} = \frac{12y+24}{13} = y + 1 + \frac{11-y}{13}.$$

Put $\frac{11-y}{13} = m$, then $y = 11 - 13m$,

and $$x = \frac{17(11-13m)+8}{13} = 15 - 17m.$$

If m be a positive integral quantity, then x and y will both be negative; if $m = 0$, then $x = 15$ and $y = 11$; if m be negative, then x and y will both be positive, and the number of positive integral solutions is unlimited.

The values of x are 15, 32, 49, 66, 83, etc.

The values of y are 11, 24, 37, 50, 63, etc.

(5.) A regiment numbers between 1600 and 1700 men. When they are divided into companies of 41 each, there are 20 left over, and when into companies of 29 each there are 7 over. How many are in the regiment?

Let N = the number

$$= 41x + 20 = 29y + 7,$$

then $$29y = 41x + 13,$$

and $$y = x + \frac{12x+13}{29}.$$

As a proper multiplier to a fraction like this may not readily be obtained, we should first multiply by any number that will leave the coefficient of x (after division by the denominator) less than at present; thus

$$3 \times \frac{12x+13}{29} = \frac{36x+39}{29} = x + 1 + \frac{7x+10}{29}.$$

It is now easy to see that multiplication by 4 will make

the coefficient of x in the numerator of the fraction differ by *one* from the denominator, and therefore we have—

$$4 \times \frac{7x+10}{29} = \frac{28x+40}{29} = x+1+\frac{11-x}{29}.$$

Putting $\frac{11-x}{29} = m$, we obtain—

$$x = 11 - 29m$$

$$\text{and } y = \frac{41(11-29m)+13}{29} = 16 - 41m.$$

$$\therefore \text{ N} = 41(11-29m)+20 = 471 - 1189m.$$

When $m = 0$, N $= 471$; when $m = -1$, N $= 1660$; and as this is within the limits stated in the question, it must be the number in the regiment.

225. If two equations containing three unknown quantities be given, we shall, by eliminating *one* of the unknowns, reduce the question to a single indeterminate equation of the same character as those just solved.

(6.) A person wishes to purchase 100 animals for £600; oxen at £21 per head, calves at £8, and sheep at £3. What number can he buy of each?

Let x, y, and z be the number of oxen, calves, and sheep respectively.

$$\text{Then } x + y + z = 100 \quad [1]$$
$$\text{and } 21x + 8y + 3z = 600 \quad [2]$$

Eliminating z, we have—

$$18x + 5y = 300.$$

Solving as in previous examples—

$$x = 5m, \text{ and } y = 60 - 18m.$$

Substituting in [1.]—

$$z = 100 - x - y = 100 - 5m - 60 + 18m = 40 + 13m.$$

As x, y, and z must each be positive and integral, we see that x will be satisfied with any integral value of m from 1 upwards, and y with any from 3 downwards, while z will admit of m being any whole number greater than -3.

The limits therefore of the values of m are 1 and 3, and the person can make his purchases in three different ways, as under :—

$$m=1,\ x=5,\ y=42,\ z=53.$$
$$m=2,\ x=10,\ y=24,\ z=66.$$
$$m=3,\ x=15,\ y=6,\ z=79.$$

The following may also be considered a solution :—

$$m=0,\ x=0,\ y=60,\ z=40.$$

In which case he purchased only calves and sheep.

226. In equations of the form $ax+by=c$, it is understood that a, b, and c are whole numbers, for if they are not, multiplication by their least common denominator will at once render them integral.

Likewise it is to be observed that if x and y are to be whole numbers, a and b cannot have a common factor not also common to c; for, suppose $a=rm$, and $b=rn$, r, m, and n being integers, then $rmx+rny=c$, and $mx+ny=\frac{c}{r}$ where, if c is not divisible by r, either x or y, or both, must be fractional.

EXAMPLES FOR PRACTICE.—LXXVI.

Find all the positive integral solutions of the following equations :—

(1.) $2x+3y=11$.
(2.) $3x+28y=149$.
(3.) $4x+7y=157$.
(4.) $12x+7y=336$.

Find the least positive integral solutions of the following :—

(5.) $5x - 13y = 48$. | (7.) $17x - 1 = 44y$.

(6.) $27x = 8y + 55$. | (8.) $29x - 66y = 213$.

Find the number of ways in which the following equations may be solved in positive integers :—

(9.) $7x + 10y = 331$. | (11.) $3x + 37y = 1225$.

(10.) $5x + 7y = 320$. | (12.) $36x + 43y = 114$.

(13.) Find all the numbers under 100 which divided by 4 leave 1, and divided by 7 leave 5.

(14.) A number of books—some at 6s. and some at 7s. —were bought for £5. How many were there of each?

(15.) In how many ways can a debt of £3, 17s. be paid in crowns and florins?

(16.) The sum of two fractions is $\frac{17}{20}$, and their denominators are 12 and 15 respectively. Find the fractions.

(17.) Tea costing 2s. 4d. per lb. was mixed with tea costing 2s. 9d. per lb., and the whole was sold for £2, 17s., bringing a profit of 6s. How many lbs. were there of each, and what was the selling price of a lb.?

(18.) A shopkeeper has only half guineas, and a customer has only crowns. Find the simplest way in which the latter can pay a shilling to the former.

(19.) Find the positive integral values of x, y, and z that satisfy the equations

$$5x - y + 4z = 114$$
$$2x - 5y + 6z = 70.$$

(20.) A purse contains £71, 10s. in guineas, crowns, and florins. There being 100 coins in all, how many are there of each sort?

(21.) A school numbers between 400 and 500 children. When they are counted by threes, there are 2 over; by fives, 4 over; and by sevens, 6 over. How many are there?

(22.) 1000 is to be divided into three parts such that

twice the first, thrice the second, and five times the third shall make up 2500. In how many ways can this be done? And when the smallest of the three parts is 101, what are the other two?

(23.) A coach, horse, and harness were bought for £232, the coach being paid for in £5 notes, the horse in guineas, and the harness in crowns. Now the horse, which cost between £80 and £90, was worth three times the price of the harness. What was paid for each?

(24.) What is the least multiple of 5 which, divided by 7, 9, or 11, always gives a remainder of 3?

CHAPTER XVII.

PERMUTATIONS AND COMBINATIONS.

227. Permutations.—The different ways in which a number of things may be arranged, one after another, are called Permutations.

If a and b be two things, either may follow the other, giving two permutations ab and ba. If a, b, and c be three things, we may arrange them in the following ways, abc, acb, bac, bca, cab, cba, six permutations; or, taking only two at a time, ab, ac, ba, bc, ca, cb, also six permutations.

In some works the name *permutations* is limited to those arrangements that include all the quantities, while the others are called *variations*.

228. PROP. I.—*To find the number of permutations that can be made with n things taken r at a time.*

First:—If n things be taken *one* at a time, they can plainly be arranged only n ways; for example, only five signals can be made with five flags when they are displayed *one* at a time.

Second:—If n things be taken two at a time, they can be arranged $n(n-1)$ ways.

Let a, b, c, d, e, etc., be n things, which are to be arranged two at a time. Remove a, then the number of things remaining will be $n-1$, and if a be placed before each, there will be $n-1$ arrangements with a first—namely, ab, ac, ad, ae, etc.

Replace a, and remove b, there will again be $n-1$ things left, and if b be placed before each, there will be $n-1$ arrangements with b first; thus, ba, bc, bd, be, etc.

Similarly there will be $n-1$ arrangements in which each of the other things, c, d, e, etc., will in turn stand first, and as there are n things in all, the total number of different arrangements must be n times $n-1$ or $n(n-1)$.

Third:—If n things be taken three at a time, they can be arranged $n(n-1)(n-2)$ ways.

Take a, b, c, d, e, etc., as before.

Remove a and b, then the number of remaining things will be $n-2$, and if ab be placed before each, so as to form an arrangement of three things, there will be $n-2$ such arrangements.

Again, if a and c be removed, there may be formed $n-2$ arrangements with three in each, and having ac first.

Similarly each of the arrangements, two at a time, may be placed in front of each of the remaining things, and in every case there will be $n-2$ arrangements of three.

As the number of arrangements, two at a time, is $n(n-1)$, the total number, three at a time, must be $n(n-1)$ times $n-2$ or $n(n-1)(n-2)$.

Fourth:—By reasoning similar to the above, we can show that n things, taken four at a time, can be arranged in $n(n-1)(n-2)(n-3)$ ways; and taken five at a time, in $n(n-1)(n-2)(n-3)(n-4)$ ways, and so on for higher numbers.

Bringing these results together, we find that the num-

ber of arrangements or permutations that can be made with n things—

One at a time $= n$.
Two at a time $= n(n-1)$.
Three at a time $= n(n-1)(n-2)$.
Four at a time $= n(n-1)(n-2)(n-3)$.
Five at a time $= n(n-1)(n-2)(n-3)(n-4)$.

Observe that the number of factors is always equal to the number of things to be taken at a time, and that the second term of the last factor is always *one less* than the number of factors, so that if r be the number of things to be taken at a time, there will be r factors, and the last one will be $n-(r-1)$ or $n-r+1$. Therefore the number of permutations of n things, taken r at a time, is $n(n-1)(n-2)\ldots$. to r factors, or, denoting the number by P_r, we have

$$P_r = n(n-1)\ldots.(n-r+1).$$

229. *Corollary.*—When n things are taken altogether, r will equal n, and $n-r+1$ will become 1; therefore the number of permutations of n things, n at a time, is $n(n-1)\ldots 3 \cdot 2 \cdot 1$.

This is generally written $\underline{|n}$, and read *factorial n*.

It follows that $\underline{|n} = n\underline{|n-1} = n(n-1)\underline{|n-2} = \text{etc.}$

Illustrative Examples.

(1.) Find the number of permutations of 11 things taken 5 at a time.

$$11 \cdot 10 \cdot 9 \cdot 8 \cdot 7 = 55440.$$

(2.) How many signals can be made with six different flags?

One at a time,		6
Two at a time,	6×5	$= 30$
Three at a time,	$6 \times 5 \times 4$	$= 120$
Four at a time,	$6 \times 5 \times 4 \times 3$	$= 360$
Five at a time,	$6 \times 5 \times 4 \times 3 \times 2$	$= 720$
Six at a time,	$6 \times 5 \times 4 \times 3 \times 2 \times 1$	$= 720$
	$\therefore$ Total number of signals	$= 1956$

(3.) If the number of permutations of n things, taken 5 together, is twenty times their number taken 3 together, how many will they form 7 together?

$$n(n-1)(n-2)(n-3)(n-4) = 20n(n-1)(n-2),$$
$$\therefore (n-3)(n-4) = 20.$$
From which $n = 8.$

And the permutations, 7 together, are

$$8 \,.\, 7 \,.\, 6 \,.\, 5 \,.\, 4 \,.\, 3 \,.\, 2 = 40320.$$

230. PROP. II.—*To find the number of permutations that can be made with n things, which are not all different, the whole being taken together.*

Of the n things let p be a's, q be b's, r be c's, and the rest all different.

Put P = the number of permutations made by the whole taken together, and suppose $a_1 \ldots . a_2 a_3 \ldots . a_4 \ldots . a_p$ to be one of them, the dots representing all the things not a's.

Then if $a_1, a_2, a_3 \ldots . a_p$ were all different from one another, and from the rest, they could make among themselves $\underline{|p}$ permutations, and as the other things could make all their permutations with each of these, the number would be increased $\underline{|p}$ times, and become $\mathrm{P}\underline{|p}$.

Similarly, if all the b's were also made different, the

number of permutations would be $P\underline{|p}\underline{|q}$; and if the c's were likewise changed, the total number would be $P\underline{|p}\underline{|q}\underline{|r}$.

But as the things are now all unlike, the total number of their permutations is $\underline{|n}$,

and $\quad \therefore P\underline{|p}\underline{|q}\underline{|r} = \underline{|n}.$

From which $\quad P = \dfrac{\underline{|n}}{\underline{|p}\underline{|q}\underline{|r}}.$

The same may be shown for any number of like things.

Illustrative Example.

How many permutations can be made with the letters of the word "Mississippi"? The number of letters is 11, of i's 4, of p's 2, of s's 4, and the rest are unlike.

$$\therefore P = \frac{\underline{|11}}{\underline{|4}\underline{|2}\underline{|4}} = \frac{11 \cdot 10 \cdot 9 \cdot 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{4 \cdot 3 \cdot 2 \cdot 1 \cdot 2 \cdot 1 \cdot 4 \cdot 3 \cdot 2 \cdot 1}$$
$$= 11 \cdot 10 \cdot 9 \cdot 7 \cdot 5 = 34650.$$

EXAMPLES FOR PRACTICE.—LXXVII.

(1.) What number of permutations can be made with 15 things, taken four at a time, and with 19, taken three at a time?

(2.) In how many ways may 7 children be seated on a form?

(3.) How many changes can be rung on eight bells? How many if only four be used each time?

(4.) What number of things, taken one or two at a time, will yield 225 permutations?

(5.) How many signals can be made with 7 flags of different colours, using them singly or in line together in every possible way?

(6.) The variations of n things, three together, are double the variations of $\frac{n}{2}$ things, four together. Find n.

(7.) How many different arrangements may be made of all the letters of each of the words, *Cataract*, *Ammunition*, *Philippic*, and *Assassination*?

(8.) If n things taken altogether make 120 permutations, what is n?

(9.) A word contains 10 letters, of which 4 are a's, a certain number are e's, and the rest are unlike. If the whole taken together make 6300 permutations, what is the number of e's?

(10.) How many permutations can be made with the factors of $ab^2c^3d^4x^2$?

(11.) If $p_1, p_2, \ldots p_{2n}$ respectively represent the permutations of $2n$ things, one at a time, two at a time, etc., to $2n$ at a time, show that $\frac{p_{2n-1}p_{2n-3}\cdot\ldots\cdot p_3p_1}{p_{2n}p_{2n-2}\cdot\ldots\cdot p_4p_2} = \frac{2^n\lfloor n}{\lfloor 2n}$

(12.) In how many ways can five children be arranged on a form so that two brothers, who are among them, may never sit together?

If n be put for the number of children, and r for the number of brothers, show that the number of ways is $\lfloor n - \lfloor r \, \lfloor n-r+1$.

231. Combinations.—The different groups, irrespective of order, into which a number of things may be formed, so many at a time, are called their combinations. Whilst permutations consist of the same things differently arranged, no two combinations can contain exactly the same things; *bca* is the same combination as *abc*.

232. PROP. I.—*To find the number of combinations that can be formed with n things taken r at a time.*

Let C_2, C_3, . . . C_r respectively represent the number of combinations that can be formed by n things, taken *two*, *three*, etc., to r at a time.

First:—Since two things taken together can form two permutations, but only one combination, when n things are taken two at a time their number of permutations must be twice their number of combinations. But their permutations, two at a time, are $n(n-1)$ [Art. 228].

$$\therefore \quad 2C_2 = n(n-1), \text{ and } C_2 = \frac{n(n-1)}{2}.$$

Second:—Since three things taken altogether can form $\lfloor 3$ or 6 permutations, but only one combination, when n things are taken three at a time their number of permutations must be six times their number of combinations. As their permutations, three at a time, are $n(n-1)(n-2)$,

$$\text{then } 6C_3 = n(n-1)(n-2), \text{ and}$$

$$C_3 = \frac{n(n-1)(n-2)}{6} \text{ or } \frac{n(n-1)(n-2)}{\lfloor 3}.$$

Third:—In a similar way we find that

$$C_4 = \frac{n(n-1)(n-2)(n-3)}{\lfloor 4} \text{ and } C_5 = \frac{n(n-1)(n-2)(n-3)(n-4)}{\lfloor 5},$$

and therefore generally that

$$C_r = \frac{n(n-1) \ldots\ldots (n-r+1)}{\lfloor r} \qquad [1]$$

$$= \frac{n(n-1) \ldots (n-r+1)(n-r) \ldots 3 \cdot 2 \cdot 1}{\lfloor r \lfloor n-r} = \frac{\lfloor n}{\lfloor r \lfloor n-r} \qquad [2]$$

233. *Corollary.*—The number of combinations of n things, taken $(n-r)$ at a time, is the same as the number taken r at a time. By [1] above

$$C_{n-r} = \frac{n(n-1)\ldots.\{n-(n-r)+1\}}{\lfloor n-r} = \frac{n(n-1)\ldots.(r+1)}{\lfloor n-r}$$

$$= \frac{n(n-1)\ldots.(r+1)r\ldots.3\,.\,2\,.\,1}{\lfloor n-r \lfloor r} = \frac{\lfloor n}{\lfloor n-r \lfloor r} = C_r.$$

234. PROP. II.—*To find how many things must be taken together so that the number of combinations may be the greatest possible.*

Let n, as before, be the number of things, and C_r the number of combinations required.

Then C_r is not $<$ either C_{r-1} or C_{r+1}.

First, $\frac{n(n-1)\ldots.(n-r+1)}{\lfloor r}$ is not $< \frac{n(n-1)\ldots.(n-r+2)}{\lfloor r-1}$

$$\therefore \quad \frac{n-r+1}{r} \text{ is not } < 1.$$

From which $n-r+1$ is not $<r$, and $\frac{n+1}{2}$ is not $<r$,

$$\text{or } r \text{ is not } > \frac{n+1}{2}. \qquad [1]$$

Second, $\frac{n(n-1)\ldots(n-r+1)}{\lfloor r}$ is not $< \frac{n(n-1)\ldots(n-r)}{\lfloor r+1}$,

$$\therefore \quad 1 \text{ is not } < \frac{n-r}{r+1}, \text{ and } r \text{ is not } < \frac{n-1}{2}. \quad [2]$$

The limiting values of r are therefore

$$\frac{n+1}{2} \text{ and } \frac{n-1}{2},$$

and since their difference is *one*, there is no intermediate value.

When n is odd, $\frac{n+1}{2}$ and $\frac{n-1}{2}$ are whole numbers, and r may equal either.

When n is even, $\frac{n+1}{2}$ and $\frac{n-1}{2}$ are fractional, and the only whole number less than the first and greater than the second is $\frac{n}{2}$.

Therefore $r = \frac{n}{2}$ when n is even,

and $\frac{n+1}{2}$, or $\frac{n-1}{2}$ when n is odd.

Illustrative Examples.

(1.) How many combinations can be made with 20 things, four at a time, and also seventeen at a time?

$$C_4 = \frac{20 \cdot 19 \cdot 18 \cdot 17}{1 \cdot 2 \cdot 3 \cdot 4} = 19 \cdot 17 \cdot 15 = 4845.$$

$$C_{17} = C_{20-17} = C_3 = \frac{20 \cdot 19 \cdot 18}{1 \cdot 2 \cdot 3} = 60 \times 19 = 1140.$$

(2.) How many different patrols of five can an officer form out of 25 men under his command? Also, how many, if he must himself always be one of the five?

$$C_5 = \frac{25 \cdot 24 \cdot 23 \cdot 22 \cdot 21}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5} = \frac{\lfloor 25}{\lfloor 5 \, \lfloor 20}.$$

When he himself goes, the others must be combined four at a time—

$$C_4 = \frac{\lfloor 25}{\lfloor 4 \, \lfloor 21}.$$

(3.) How many different sums can be paid with a guinea, a sovereign, a half sovereign, a crown, a florin, a shilling, a sixpence, and a penny?

There are here eight pieces :—

$$C_1 = 8$$

$$C_2 = \frac{8 \cdot 7}{1 \cdot 2} = 28$$

$$C_3 = \frac{8 \cdot 7 \cdot 6}{1 \cdot 2 \cdot 3} = 56$$

$$C_4 = \frac{8 \cdot 7 \cdot 6 \cdot 5}{1 \cdot 2 \cdot 3 \cdot 4} = 70$$

$$C_5 = C_3 = 56$$

$$C_6 = C_2 = 28$$

$$C_7 = C_1 = 8$$

$$C_8 = 1$$

$$\therefore \text{ Total number} = 255$$

EXAMPLES FOR PRACTICE.—LXXVIII.

(1.) What number of combinations can be made with 15 things, taken four at a time, and with 19, three at a time?

(2.) Find the number of combinations that can be formed out of 21 things, fourteen at a time, and out of 100 things, ninety-six at a time?

(3.) What is the difference between the number of combinations of 16 things, taken five at a time, and 15 things, taken six at a time?

(4.) A person wishes to make up as many different dinner parties as possible from among 21 friends. How many should he invite at a time?

(5.) From a company of 40 soldiers, four are placed on guard every two hours. For what length of time can a different set be selected?

(6.) How many different sums can be paid with a farthing, a halfpenny, a penny, a sixpence, a shilling, a florin, a half-crown, a crown, and a sovereign?

(7.) Of ten balls of different colours, one is white. If they be formed into as many sets of four as possible, in how many of these will the white one be found?

(8.) Let C_r^n represent the combinations of n things r at a time. Given $C_4^n : C_2^n :: 5 : 3$, to find n.

(9.) If $C_5^{n+1} = 7C_3^n$, what is the value of n?

(10.) Let P_r^n represent the permutations of n things r at a time. Given $P_r^n = 42P_{r-1}^n$ and $C_r^n = C_{r-1}^n$, to find n and r.

(11.) Show that $\dfrac{C_{2n}^{4n}}{C_n^{2n}} = \dfrac{\lfloor 4n (\lfloor n)^2}{(\lfloor 2n)^3}$.

(12.) Prove that $C_r^n + C_{r-1}^n = C_r^{n+1}$.

CHAPTER XVIII.

BINOMIAL THEOREM.

235. When any power of a binomial quantity is expanded into a series, the law which determines the successive terms is known as the Binomial Theorem.

236. Preliminary to giving a demonstration of the theorem, it will be necessary to establish the following

PROPOSITION. — If two expressions containing one or more variable quantities are identical (Art. 14), the coefficients of the like powers of like variables are equal.

First, let $a_0 + a_1x + a_2x^2 + \ldots = b_0 + b_1x + b_2x^2 + \ldots$ be an identity.

Since the two sides are equal for all values of x, put

$$x = 0,$$

Then $$a_0 = b_0$$

And $$a_1x + a_2x^2 + \ldots = b_1x + b_2x^2 + \ldots.$$

Dividing by x, $$a_1 + a_2x + \ldots = b_1 + b_2x + \ldots$$

And again putting $$x = 0$$

Then $$a_1 = b_1.$$

Similarly we may prove $a_2 = b_2$, and so on for the other coefficients.

Second, let the following be an identity :—

$$\left.\begin{array}{l} a_0+a_1x+a_2x^2+\ldots \\ +a'_0y+a'_1xy+a'_2x^2y+\ldots \\ +a''_0y^2+a''_1xy^2+a''_2x^2y^2+\ldots \\ +\text{etc.}\ldots\ldots\ldots\ldots \end{array}\right\} = \left\{\begin{array}{l} b_0+b_1x+b_2x^2+\ldots \\ +b'_0y+b'_1xy+b'_2x^2y+\ldots \\ +b''_0y^2+b''_1xy^2+b''_2x^2y^2+\ldots \\ +\text{etc.}\ldots\ldots\ldots\ldots \end{array}\right.$$

Since x and y may have any value, consider x constant, then $a_0+a_1x+a_2x^2+\ldots$, $a'_0+a'_1x+a'_2x^2+\ldots$, etc., will also be constant, and the given identity may be written under the form

$$A_0+A_1y+A_2y^2+\ldots=B_0+B_1y+B_2y^2+\ldots$$

As in the first case, we can show that

$$A_0=B_0,\ A_1=B_1,\ A_2=B_2,\ \text{etc.},$$

$$\begin{array}{ll} \text{and } \therefore & a_0+a_1x+a_2x^2+\ldots=b_0+b_1x+b_2x^2+\ldots \\ & a'_0+a'_1x+a'_2x^2+\ldots=b'_0+b'_1x+b'_2x^2+\ldots \\ & a''_0+a''_1x+a''_2x^2+\ldots=b''_0+b''_1x+b''_2x^2+\ldots \\ & \text{etc.} \qquad\qquad =\text{etc.} \end{array}$$

Now consider x variable, and, as before, we have—

$$\begin{array}{lll} a_0=b_0, & a_1=b_1, & a_2=b_2, \text{ etc.} \\ a'_0=b'_0, & a'_1=b'_1, & a'_2=b'_2, \text{ etc.} \\ a''_0=b''_0, & a''_1=b''_1, & a''_2=b''_2, \text{ etc.} \\ \text{etc.} & \text{etc.} & \text{etc.} \end{array}$$

237. Positive Integral Exponent.—When we have given the binomials, $x+a_1$, $x+a_2$, $x+a_3$, etc., by multiplication we can easily obtain the following :—

$$\begin{aligned} (x+a_1)(x+a_2) &= x^2+(a_1+a_2)x+a_1a_2 \\ (x+a_1)(x+a_2)(x+a_3) &= x^3+(a_1+a_2+a_3)x^2 \\ &\quad +(a_1a_2+a_1a_3+a_2a_3)x+a_1a_2a_3 \\ (x+a_1)(x+a_2)(x+a_3)(x+a_4) &= x^4+(a_1+a_2+a_3+a_4)x^3 \\ &\quad +(a_1a_2+a_1a_3+a_1a_4+a_2a_3+a_2a_4+a_3a_4)x^2 \\ &\quad +(a_1a_2a_3+a_1a_2a_4+a_1a_3a_4+a_2a_3a_4)x+a_1a_2a_3a_4. \end{aligned}$$

In these successive products observe—

1st. That the number of terms is one more than the number of factors.

2nd. That the exponent of x in the first term is equal to the number of factors.

3rd. That the exponent of x decreases by *one* in each succeeding term.

4th. That the coefficient of the first term is *one*; the coefficient of the second term is the sum of the second terms of the factors; that of the third term is the sum of the products of the second terms of the factors taken two at a time; that of the fourth term is the sum of the products of the same quantities taken three at a time; and so on.

5th. That the last term is the product of the second terms of the factors taken altogether.

If now we suppose all the a's to be the same, and that there are n of them, then

$(x+a_1)(x+a_2)\ldots(x+a_n)$ will become $(x+a)^n$,

$a_1+a_2+\ldots a_n$ will equal na,

$a_1a_2+a_1a_3+\ldots a_{n-1}a_n$ will equal $\frac{n(n-1)}{2}a^2$,

$a_1a_2a_3+\ldots a_{n-2}a_{n-1}a_n$ will equal $\frac{n(n-1)(n-2)}{1\,.\,2\,.\,3}a^3$,

. . . . . .

and $a_1a_2\ldots a_n$ will equal a^n.

From which it appears that after the first term, the successive numerical coefficients are the same as the combinations of n things taken one at a time, two at a time, to n at a time.

Assuming this to be true, we may write—

$$(x+a)^n = x^n + nax^{n-1} + \frac{n(n-1)}{2}a^2x^{n-2} + \frac{n(n-1)(n-2)}{1\,.\,2\,.\,3}a^3x^{n-3} + \ldots + na^{n-1}x + a^n. \qquad [\text{I.}]$$

Next, multiply both sides by $x+a$, then

$$x(x+a)^n = x^{n+1} + nax^n + \frac{n(n-1)}{2}a^2x^{n-1} + \ldots + a^n x$$
$$a(x+a)^n = \quad + ax^n + na^2x^{n-1} + \ldots + na^n x + a^{n+1}$$

$$\therefore (x+a)^{n+1} = x^{n+1} + (n+1)ax^n + \frac{(n+1)n}{2}a^2x^{n-1} + \ldots + (n+1)a^n x + a^{n+1},$$

or, putting $n+1=m$,

$$(x+a)^m = x^m + max^{m-1} + \frac{m(m-1)}{2}a^2x^{m-2} + \ldots + ma^{m-1}x + a^m,$$

which is of the same form as I., and shows that if the law of expansion in I. is true for the exponent n, it is also true for the exponent $n+1$. But in Art. 133 it was shown to be true for $n=6$; it is therefore true for $n=7$, therefore also for $n=8$, and so on. Wherefore, when n is a whole number, the expansion of $(x+a)^n$ in I. is always true.

In the expansion of $(x-a)^n$, the numerical coefficients, depending only on n and their position in the series, will evidently be the same as those of $(x+a)^n$, while the signs of the terms containing the odd powers of a will be negative:—

$$\therefore (x-a)^n = x^n - nax^{n-1} + \frac{n(n-1)}{2}a^2x^{n-2} - \ldots + (-a)^n.$$

238. Fractional Exponent.—Since $(a+x) = a\left(1+\frac{x}{a}\right)$, $(a+x)^n$ may be written $a^n\left(1+\frac{x}{a}\right)^n = a^n(1+x')^n$, if $x' = \frac{x}{a}$. The expansion of $(a+x)^n$ may therefore be made to depend on that of $(1+x')^n$.

Now let it be required to find the expansion of $(1+x)^{\frac{p}{q}}$, $\frac{p}{q}$ being any positive fraction.

Assume $(1+x)^{\frac{p}{q}} = 1 + Ax + Bx^2 + Cx^3 + Dx^4 + \text{ etc.}$, [1]

the first term of the expansion being 1, because when $x = 0$, each side will be equal to 1.

Also, assume $(1+y)^{\frac{p}{q}} = 1 + Ay + By^2 + Cy^3 + Dy^4 + \text{etc.}$, [2]

the coefficients being the same as in [1], since they do not depend for their value on x or y.

Put $u^p = (1+x)^{\frac{p}{q}}$, and $v^p = (1+y)^{\frac{p}{q}}$,

then $u^q = 1 + x$, and $v^q = 1 + y$ [3]

$\therefore\ u^q - v^q = x - y$ [4]

From [1], $u^p = 1 + Ax + Bx^2 + Cx^3 + Dx^4 + \text{etc.}$ [5]

From [2], $v^p = 1 + Ay + By^2 + Cy^3 + Dy^4 + \text{etc.}$ [6]

Taking [6] from [5]—

$$u^p - v^p = A(x-y) + B(x^2-y^2) + C(x^3-y^3) + D(x^4-y^4) + \text{etc.} \quad [7]$$

Dividing [7] by [4]—

$$\frac{u^p - v^p}{u^q - v^q} = \frac{A(x-y) + B(x^2-y^2) + C(x^3-y^3) + D(x^4-y^4) + \text{etc.}}{x-y}$$

Dividing by the common factors $u - v$ and $x - y$,

$$\frac{u^{p-1} + u^{p-2}v + \ldots + v^{p-1}}{u^{q-1} + u^{q-2}v + \ldots + v^{q-1}} = A + B(x+y) + C(x^2 + xy + y^2) + D(x^3 + x^2y + xy^2 + y^3) + \text{etc.} \quad [8]$$

If now $x = y$, then $u^p = (1+x)^{\frac{p}{q}} = (1+y)^{\frac{p}{q}} = v^p$,

and $u = v$.

Substituting in [8], the left hand side becomes

$$\frac{u^{p-1} + u^{p-1} + u^{p-1} + \ldots \text{ to } p \text{ terms}}{u^{q-1} + u^{q-1} + \ldots \text{ to } q \text{ terms}} = \frac{p}{q} \cdot \frac{u^{p-1}}{u^{q-1}} = \frac{p}{q} \cdot \frac{u^p}{u^q},$$

by multiplying by $\frac{u}{u}$.

Also the right hand side becomes

$$A + 2Bx + 3Cx^2 + 4Dx^3 + \text{etc.}$$

$$\therefore \frac{p}{q} \cdot \frac{u^p}{u^q} = A + 2Bx + 3Cx^2 + 4Dx^3 + \text{etc.} \qquad [9]$$

From [3], $u^q = 1 + x$; from [5], $u^p = 1 + Ax + Bx^2 + Cx^3 + \text{etc.}$

$$\therefore \frac{p}{q} \cdot \frac{1 + Ax + Bx^2 + Cx^3 + \text{etc.}}{1 + x} = A + 2Bx + 3Cx^2 + 4Dx^3 + \text{etc.} \quad [10]$$

Multiplying by $1 + x$, then

$$\frac{p}{q}(1 + Ax + Bx^2 + Cx^3 + \text{etc.}) = A + (A + 2B)x + (2B + 3C)x^2 + (3C + 4D)x^3 + \text{etc.}$$

Now equating coefficients of like powers of x (Art. 236), we get—

$$A = \frac{p}{q}, \; B = \frac{A}{2}\left(\frac{p}{q} - 1\right) = \frac{1}{2}\frac{p}{q}\left(\frac{p}{q} - 1\right)$$

$$C = \frac{B}{3}\left(\frac{p}{q} - 2\right) = \frac{1}{2} \cdot \frac{1}{3}\frac{p}{q}\left(\frac{p}{q} - 1\right)\left(\frac{p}{q} - 2\right)$$

$$D = \frac{C}{4}\left(\frac{p}{q} - 3\right) = \frac{1}{2} \cdot \frac{1}{3} \cdot \frac{1}{4}\frac{p}{q}\left(\frac{p}{q} - 1\right)\left(\frac{p}{q} - 2\right)\left(\frac{p}{q} - 3\right),$$

and similarly for the other coefficients.

$$\therefore (1+x)^{\frac{p}{q}} = 1 + \frac{p}{q}x + \frac{\frac{p}{q}\left(\frac{p}{q} - 1\right)}{2}x^2 + \frac{\frac{p}{q}\left(\frac{p}{q} - 1\right)\left(\frac{p}{q} - 2\right)}{2 \cdot 3}x^3 + \text{etc.} \quad [\text{II.}]$$

239. When the exponent is $\frac{-p}{q}$, the following modification of the foregoing investigation will be necessary :—

$$\text{Since } u^{-p} - v^{-p} = \frac{1}{u^p} - \frac{1}{v^p} = -\frac{u^p - v^p}{u^p v^p},$$

equation [7] will become

$$-\frac{u^p - v^p}{u^p v^p} = A(x - y) + B(x^2 - y^2) + C(x^3 - y^3) + \text{etc.},$$

from which, by division, we derive—

$$-\frac{1}{u^p v^p}\cdot\frac{u^{p-1}+u^{p-2}v+\ldots+v^{p-1}}{u^{q-1}+u^{q-2}v+\ldots v^{q-1}}=\mathrm{A}+\mathrm{B}(x+y)+\mathrm{C}(x^2+xy+y^2)+\text{etc.}$$

And making, as before, $x=y$, and $\therefore\ u=v$, we have—

$$-\frac{1}{u^{2p}}\cdot\frac{p}{q}\cdot\frac{u^{p-1}}{u^{q-1}}=-\frac{p}{q}\cdot\frac{u^{-p}}{u^q}=\mathrm{A}+2\mathrm{B}x+3\mathrm{C}x^2+\text{etc.}$$

But $u^{-p}=1+\mathrm{A}x+\mathrm{B}x^2+$ etc., and $u^q=1+x$,

$$\therefore\ -\frac{p}{q}\cdot\frac{1+\mathrm{A}x+\mathrm{B}x^2+\text{etc.}}{1+x}=\mathrm{A}+2\mathrm{B}x+3\mathrm{C}x^2+\text{etc.},$$

and $$-\frac{p}{q}(1+\mathrm{A}x+\mathrm{B}x^2+\text{etc.})=\mathrm{A}+(\mathrm{A}+2\mathrm{B})x+(2\mathrm{B}+3\mathrm{C})x^2+\text{etc.}$$

Wherefore $\mathrm{A}=-\frac{p}{q}$; $\mathrm{B}=\frac{\frac{p}{q}\left(\frac{p}{q}+1\right)}{2}$; $\mathrm{C}=-\frac{\frac{p}{q}\left(\frac{p}{q}+1\right)\left(\frac{p}{q}+2\right)}{2\,.\,3}$,

etc.

$$\therefore\ (1+x)^{-\frac{p}{q}}=1-\frac{p}{q}x+\frac{\frac{p}{q}\left(\frac{p}{q}+1\right)}{2}x^2-\frac{\frac{p}{q}\left(\frac{p}{q}+1\right)\left(\frac{p}{q}+2\right)}{2\,.\,3}x^3+\text{etc.}\quad[\text{III.}]$$

As the terms of the expansion of $1-x$ to any positive power would be alternately positive and negative (Art. 237), then

$$(1-x)^{-\frac{p}{q}}=1+\frac{p}{q}x+\frac{\frac{p}{q}\left(\frac{p}{q}+1\right)}{2}x^2+\frac{\frac{p}{q}\left(\frac{p}{q}+1\right)\left(\frac{p}{q}+2\right)}{2\,.\,3}x^3+\text{etc.}\quad[\text{IV.}]$$

If $-\frac{p}{q}$ be represented by n, then in III. and IV.

$$(1+x)^{-\frac{p}{q}}=(1+x)^n=1+nx+\frac{n(n-1)}{2}x^2+\frac{n(n-1)(n-2)}{2\,.\,3}x^3+\text{etc.}$$

and

$$(1-x)^{-\frac{p}{q}}=(1-x)^n=1-nx+\frac{n(n-1)}{2}x^2-\frac{n(n-1)(n-2)}{2\,.\,3}x^3+\text{etc.}$$

So that whether n be whole or fractional, the theorem always holds true that

$$(1 \pm x)^n = 1 \pm nx + \frac{n(n-1)}{2}x^2 \pm \frac{n(n-1)(n-2)}{2 \cdot 3}x^3 + \text{etc.}$$

Putting $\frac{x}{a}$ for $\pm x$, then

$$\left(1 + \frac{x}{a}\right)^n = 1 + n\frac{x}{a} + \frac{n(n-1)}{2}\frac{x^2}{a^2} + \frac{n(n-1)(n-2)}{2 \cdot 3}\frac{x^3}{a^3} + \text{etc.},$$

and $(a+x)^n = a^n\left(1 + \frac{x}{a}\right)^n$

$$= a^n\left\{1 + n\frac{x}{a} + \frac{n(n-1)}{2} \cdot \frac{x^2}{a^2} + \frac{n(n-1)(n-2)}{2 \cdot 3}\frac{x^3}{a^3} + \text{etc.}\right\}$$

$$= a^n + na^{n-1}x + \frac{n(n-1)}{2}a^{n-2}x^2 + \frac{n(n-1)(n-2)}{2 \cdot 3}a^{n-3}x^3 + \text{etc.} \quad [\text{V.}]$$

240. General Term.—In the above expression, observe that the exponents of a and x in any term together amount to n, and that the exponent of x is always *one* less than the number of the term in which it stands. Wherefore in the rth term the exponent of x is $(r-1)$, and of a $(n-r+1)$.

Observe also that the numerical exponent of the rth term is $\frac{n(n-1)\ldots(n-r+2)}{1 \cdot 2 \cdot 3 \ldots (r-1)}$, for in every case the right-hand factor of the denominator is the same as the exponent of x, and the right-hand factor of the numerator is one more than the exponent of a.

We have therefore the following general expression for any term of the series, excepting the first, r denoting the number of the term :—

$$\frac{n(n-1)(n-2)\ldots(n-r+2)}{1 \cdot 2 \cdot 3 \ldots (r-1)}a^{n-r+1}x^{r-1} \quad [\text{VI.}]$$

If multiplied by $\frac{\lfloor n-r+1}{\lfloor n-r+1}$, this may be written—

$$\frac{\lfloor n}{\lfloor r-1 \; \lfloor n-r+1} a^{n-r+1}x^{r-1} \qquad \text{[VII.]}$$

Note.—This expression will include the first term if $\lfloor 0$ be considered equal to 1.

241. *When n is a whole number, the coefficient* $\frac{n(n-1)(n-2)\ldots(n-r+2)}{1\,.\,2\,.\,3\ldots(r-1)}$ *is also a whole number.*

Since the product of any two consecutive numbers is divisible by 2, of any three by 2 and 3, of any four by 2, 3, and 4, and so on, the above numerator being the product of $r-1$ consecutive numbers, is necessarily divisible by $\lfloor r-1$.

242. *When n is a positive integer, the number of terms is* $n+1$.

If r be made equal to any number up to $n+1$, the coefficient of the general term will be found to have an integral value; but if r be made equal to any higher number, as $n+2$, $n+3$, etc., the coefficient will become nothing, since it will contain the factor $n-r+2=0$. The number of terms is therefore $n+1$, and no more.

Corollary.—When n is negative or fractional, the number of terms is unlimited, for r being integral and positive, no possible value of it can make $n-r+2=0$, and therefore 0 does not enter as a factor into any coefficient of the expansion.

243. *When n is a positive integer, the coefficient of the rth term from the end is equal to that of the rth term from the beginning.*

Since the total number of terms is $n+1$ (Art. 242), the rth from the end will be the $(n-r+2)$th from the beginning, and its coefficient will be—

$$\frac{n(n-1)\ldots\{n-(n-r+2)+2\}}{1\,.\,2\,.\,3\ldots(n-r+1)}$$

$$=\frac{n(n-1)(n-2)\ldots r}{1\,.\,2\,.\,3\ldots(n-r+1)}$$

$$=\frac{\lfloor n}{\lfloor n-r+1 \,\lfloor r-1},$$

which is also the coefficient of the rth term from the beginning (Art. 240).

This proposition may be made useful in expanding any binomial having a positive integral index, for the coefficients after the middle being the same in reverse order as those before it, may be at once written down without calculation. When the number of terms is odd, there will be one middle term; when even, two; and as in this last case the first is as far from the beginning as the second is from the end, their coefficients must be equal.

. The number of terms will be odd or even according as n is even or odd (Art. 242).

244. The Greatest Coefficient.—From [V.] and [VI.], Arts. 239 and 240, the coefficient of the rth term is $\frac{n-r+2}{r-1}$ times that of the previous one.

So long as $\frac{n-r+2}{r-1}$ is > 1, the successive coefficients will increase, and when $\frac{n-r+2}{r-1}$ is < 1, they will diminish. Now, n and r being positive integers, as r becomes larger, $\frac{n-r+2}{r-1}$ must become smaller, and eventually < 1. The coefficients will, therefore, form a series increasing up

to a certain point, and then decreasing to the end. As by last article the coefficients of the first part of the expression reckoned from the beginning are identical with those of the second part reckoned from the end, the greatest coefficient must be that of the middle term when the number of terms is odd, or of either of the two middle terms when the number of terms is even :—

That is, when $r = \frac{n+2}{2}$ for n even ;

And when $r = \frac{n+1}{2}$ or $\frac{n+3}{2}$ for n odd.

245. The Greatest Term.—From Arts. 239 and 240, the $(r+1)$th term is $\frac{n-r+1}{r} \cdot \frac{x}{a}$ times the rth. So long as $\frac{n-r+1}{r} \cdot \frac{x}{a}$ is > 1, the terms will increase; but when $\frac{n-r+1}{r} \cdot \frac{x}{a}$ is < 1, they will diminish. The rth term will therefore be greatest when $\frac{n-r+1}{r} \cdot \frac{x}{a} = 1$, or is first < 1, that is, when $(n-r+1)x = ar$, or is first $< ar$, that is, when $(n+1)x = (a+x)r$, or is first $< (a+x)r$, that is, when $r = \frac{(n+1)x}{a+x}$, or is first $> \frac{(n+1)x}{a+x}$.

The greatest term is, therefore, the one whose place is indicated by $\frac{(n+1)x}{a+x}$, or the first whole number $> \frac{(n+1)x}{a+x}$.

When $\frac{(n+1)x}{a+x}$ is a whole number, there are two equal terms greater than any other.

In the expansion of $(a \pm x)^{-n}$, n being a whole number, the greatest term may, in similar manner, be found to be indicated by $\frac{(n-1)x}{a-x}$.

246. The Sum of the Numerical Coefficients.

In the expression

$$(a+x)^n = a^n + na^{n-1}x + \frac{n(n-1)}{1 \cdot 2} a^{n-2}x^2 + \text{etc.},$$

let a and x each equal 1, then

$$(1+1)^n = 2^n = 1 + n + \frac{n(n-1)}{1 \cdot 2} + \frac{n(n-1)(n-2)}{1 \cdot 2 \cdot 3} + \text{ etc.},$$

that is, the sum of the numerical coefficients $= 2^n$.

247. The Sum of the Numerical Coefficients of the alternate terms.

In the expression

$$(a-x)^n = a^n - na^{n-1}x + \frac{n(n-1)}{1 \cdot 2} a^{n-2}x^2 - \text{ etc.},$$

let a and x each equal 1, then

$$(1-1)^n = 0 = 1 - n + \frac{n(n-1)}{1 \cdot 2} - \frac{n(n-1)(n-2)}{1 \cdot 2 \cdot 3} + \text{ etc.}$$

$$\therefore \ 1 + \frac{n(n-1)}{1 \cdot 2} + \text{ etc.} = n + \frac{n(n-1)(n-2)}{1 \cdot 2 \cdot 3} + \text{ etc.}$$

That is, the sum of the coefficients of the odd terms is equal to the sum of the coefficients of the even terms, and since the sum of the whole is 2^n, that of the coefficients of the alternate terms must be $\frac{2^n}{2}$ or 2^{n-1}.

248. Applications of Binomial Theorem.

Illustrative Examples.

(1.) Expand $(1-x)^{-\frac{2}{3}}$ to five terms.

$$(1-x)^{-\frac{2}{3}} = 1^{-\frac{2}{3}} + \frac{2}{3}(1)^{-\frac{5}{3}}x + \frac{5}{9}(1)^{-\frac{8}{3}}x^2 + \frac{40}{81}(1)^{-\frac{11}{3}}x^3 + \frac{110}{243}(1)^{-\frac{14}{3}}x^4 + \text{etc.}$$

$$= 1 + \frac{2}{3}x + \frac{5}{9}x^2 + \frac{40}{81}x^3 + \frac{110}{243}x^4 + \text{etc.}$$

(2.) Write down the sixth term of the expansion of $\left(2-\frac{x}{3}\right)^{\frac{9}{4}}$, and say which is the greatest term when $x=20$.

Substitute in the general term, noticing that the even terms are minus.

$$n=\frac{9}{4} \quad r=6, \therefore n-r+1=\frac{9}{4}-5=-\frac{11}{4}.$$

$$\therefore \text{ 6th term} = -\frac{\frac{9}{4}\left(\frac{9}{4}-1\right)\left(\frac{9}{4}-2\right)\left(\frac{9}{4}-3\right)\left(\frac{9}{4}-4\right)}{1.2.3.4.5} \cdot 2^{-\frac{11}{4}} \cdot \frac{x^5}{3^5}$$

$$= -\frac{9}{4}\cdot\frac{5}{4}\cdot\frac{1}{4}\cdot\left(-\frac{3}{4}\right)\cdot\left(-\frac{7}{4}\right)\cdot\frac{1}{1.2.3.4.5}\cdot\frac{1}{2^{11}}\cdot\frac{x^5}{3^5}$$

$$= -\frac{7 \cdot 2^{\frac{1}{4}}}{2^{16} \cdot 3^3}x^5.$$

$$\text{Also } \frac{(n+1)x}{a+x} = \frac{\left(\frac{9}{4}+1\right)\frac{20}{3}}{2+\frac{20}{3}} = \frac{13}{4}\cdot\frac{20}{26} = 2\tfrac{1}{2},$$

$\therefore$ the third term is the greatest.

(3.) Expand $\frac{3+2x}{4+3x}$ in a series of ascending powers of x.

$$\frac{3+2x}{4+3x} = (3+2x)(4+3x)^{-1} = (3+2x)\left\{4^{-1} - 1\,.\,4^{-2}(3x)\right.$$

$$\left. + \frac{1.2}{1.2}\cdot 4^{-3}(3x)^2 - \frac{1.2.3}{1.2.3}\cdot 4^{-4}(3x)^3 + \frac{1.2.3.4}{1.2.3.4}\cdot 4^{-5}(3x)^4 - \text{etc.}\right\}$$

$$= \frac{3}{4} - \left(\frac{9}{4^2} - \frac{2}{4}\right)x + \left(\frac{27}{4^3} - \frac{6}{4^2}\right)x^2 - \left(\frac{81}{4^4} - \frac{18}{4^3}\right)x^3 + \left(\frac{243}{4^5} - \frac{54}{4^4}\right)x^4 - \text{etc.}$$

$$= \frac{3}{4} - \frac{1}{4^2}x + \frac{3}{4^3}x^2 - \frac{9}{4^4}x^3 + \frac{27}{4^5}x^4 - \text{etc.}$$

(4.) Find approximately the third root of 225.

$$\sqrt[3]{225} = (216+9)^{\frac{1}{3}} = 216^{\frac{1}{3}} + \frac{1}{3}\cdot 216^{-\frac{2}{3}}\cdot 9 - \frac{1}{9}\cdot 216^{-\frac{5}{3}}\cdot 9^2$$

$$+ \frac{5}{9^2}\cdot 216^{-\frac{8}{3}}\cdot 9^3 - \frac{10}{3\,.\,9^2}\cdot 216^{-\frac{11}{3}}\cdot 9^4 + \text{etc.}$$

$$= 6 + \frac{3}{6^2} - \frac{9}{6^5} + \frac{45}{6^8} - \frac{270}{6^{11}} + \text{etc.}$$

$$= 6 + \cdot 0833333 - \cdot 0011574 + \cdot 0000267 - \cdot 0000007 + \text{etc.}$$

$$= 6\cdot 0822019 + \text{etc.}$$

(5.) Find the sum of the squares of the coefficients in the expansion of $(1+x)^n$, n being a positive integer.

Put $(1+x)^n = A + Bx + Cx^2 + \ldots + Cx^{n-2} + Bx^{n-1} + Ax^n$,

also $(1+x)^n = Ax^n + Bx^{n-1} + Cx^{n-2} + \ldots + Cx^2 + Bx + A$.

Multiply these lines together, and on second side write first those terms that contain x^n,

then $(1+x)^{2n} = A^2x^n + B^2x^n + C^2x^n + \ldots + C^2x^n + B^2x^n$

$$+ A^2x^n + \text{etc.} \ldots \text{etc.} \ldots \text{etc.}$$

$$= (A^2 + B^2 + C^2 + \ldots + C^2 + B^2 + A^2)x^n + \text{etc.}$$

But the coefficient of x^n in the expansion of $(1+x)^{2n}$ is

$$\frac{2n(2n-1)\ldots\{2n-(n+1)+2\}}{1\cdot 2\ldots n}$$

$$=\frac{2n(2n-1)\ldots(n+1)}{1\cdot 2\ldots n}$$

$$=\frac{\lfloor 2n}{\lfloor n\ \lfloor n}\cdot \quad \left(\text{Multiplying by } \frac{\lfloor n}{\lfloor n}\right).$$

$$\therefore \text{ by Art. 236, } A^2+B^2+C^2+\text{etc.}=\frac{\lfloor 2n}{\lfloor n\ \lfloor n}.$$

(6.) If $X_r=x(x+1)(x+2)\ldots(x+r-1)$,
and $Y_r=y(y+1)(y+2)\ldots(y+r-1)$,

then $(X+Y)_r=X_r+rX_{r-1}Y+\frac{r(r-1)}{1\cdot 2}X_{r-2}Y_2+\ldots+Y_r$

Expand $(1+z)^X$ and $(1+z)^Y$,

then $(1+z)^X=1+Xz+\frac{X_2}{1\cdot 2}z^2+\ldots+\frac{X_{r-1}}{\lfloor r-1}z^{r-1}+\frac{X_r}{\lfloor r}z^r+\text{etc.}$

and $(1+z)^Y=1+Yz+\frac{Y_2}{1\cdot 2}z^2+\ldots+\frac{Y_{r-1}}{\lfloor r-1}z^{r-1}+\frac{Y_r}{\lfloor r}z^r+\text{etc.}$

Multiply, and write first the terms containing z^r.

$$(1+z)^{X+Y}=\left(\frac{X_r}{\lfloor r}+\frac{X_{r-1}}{\lfloor r-1}\cdot Y+\frac{X_{r-2}}{\lfloor r-2}\cdot\frac{Y_2}{1\cdot 2}+\ldots+\frac{Y_r}{\lfloor r}\right)z^r+\text{etc.}$$

But $(1+z)^{X+Y}=1+(X+Y)z+\ldots\ldots+\frac{(X+Y)^r}{\lfloor r}z^r+\text{etc.}$

Equate coefficients of z^r,

then $\frac{(X+Y)_r}{\lfloor r}=\frac{X_r}{\lfloor r}+\frac{X_{r-1}}{\lfloor r-1}Y+\frac{X_{r-2}}{\lfloor r-2}\cdot\frac{Y_2}{1\cdot 2}+\ldots+\frac{Y_r}{\lfloor r}.$

Multiply by $\lfloor r$.

$$\therefore\ (X+Y)_r=X_r+rX_{r-1}Y+\frac{r(r-1)}{1\cdot 2}X_{r-2}Y_2+\ldots\ldots+Y_r.$$

EXAMPLES FOR PRACTICE.—LXXIX.

(1.) Expand $(3-x^2)^9$.

(2.) Find the eighth term of $(2+x)^8$, and the sixth of $(a-3)^{11}$.

(3.) Expand $\left(\frac{2}{3}-\frac{1}{2}x\right)^6$.

(4.) Find the middle term in the expansion of $\left(\frac{1}{2}+2x^2\right)^{10}$, and the coefficient of x^6.

(5.) Expand $(1+3x)^{-3}$ to four terms.

(6.) Write the sixteenth term of $(1-x)^{-1}$ and of $(1+x^2)^{-2}$.

(7.) Expand $(1+4x)^{\frac{11}{2}}$ to five terms.

(8.) Find the term that does not contain x in the expansion of $\left(x^3-\frac{1}{x}\right)^{\frac{8}{3}}$.

(9.) Expand $(1-3x)^{\frac{2}{3}}$ to six terms.

(10.) Find the rth term in the expansion of $(1+x)^9$, and of $(1-4x)^{\frac{1}{2}}$.

(11.) Expand $(a^2+x^2)^{-\frac{3}{4}}$ to five terms.

(12.) What are the greatest terms in the expansion of $\left(\frac{3}{4}+2x^2\right)^{10}$, when $x=1$, and when $x=\frac{1}{2}$?

(13.) Expand $\frac{\sqrt{a}+\sqrt{x}}{(\sqrt{a}-\sqrt{x})^2}$ to six terms.

(14.) Find the general term in the expansion of $\left(1-\frac{x}{m}\right)^{-m}$ and of $\frac{1}{\sqrt[5]{1+5x}}$.

(15.) Find approximately the sixth root of 730, and the tenth root of 1000.

(16.) After which terms do the expansions of $\left(1-\frac{3}{7}\right)^{-5}$ and $\left(1+\frac{5}{3}\right)^{-5}$ begin to decrease?

(17.) Show that $\sqrt{2}=1+\frac{1}{4}+\frac{1\cdot 3}{4\cdot 8}+\frac{1\cdot 3\cdot 5}{4\cdot 8\cdot 12}+$ etc. to infinity.

(18.) What is the first term with a negative coefficient in the expansion of $(1+2x)^{\frac{1}{2}}$?

(19.) If in the expansion of $(a+x)^n$, S_1 be the sum of the odd terms, and S_2 the sum of the even terms, show that $S_1^2-S_2^2=(a^2-x^2)^n$ and $4S_1S_2=(a+x)^{2n}-(a-x)^{2n}$.

(20.) Find the general term of $1+\frac{1}{3}x+\frac{1\cdot 3}{3\cdot 6}x^2+\frac{1\cdot 3\cdot 5}{3\cdot 6\cdot 9}x^3+$ etc., and its sum to infinity when $x=1$.

(21.) Prove that the coefficient of the $(r+1)$th term of $(1+x)^{n+1}$ is equal to the sum of the coefficients of the rth and $(r+1)$th terms of $(1+x)^n$.

(22.) Show that

$$1+n\left(\frac{2n}{1-n}\right)+\frac{n(n-1)}{1\cdot 2}\left(\frac{2n}{1-n}\right)^2+\frac{n(n-1)(n-2)}{1\cdot 2\cdot 3}\left(\frac{2n}{1-n}\right)^3+\text{etc.}$$

is equal to

$$1+n\left(\frac{2n}{1+n}\right)+\frac{n(n+1)}{1\cdot 2}\left(\frac{2n}{1+n}\right)^2+\frac{n(n+1)(n+2)}{1\cdot 2\cdot 3}\left(\frac{2n}{1+n}\right)^3+\text{etc.}$$

(23.) Find the sum of

$$1+2n+3\frac{n(n-1)}{1\cdot 2}+4\frac{n(n-1)(n-2)}{1\cdot 2\cdot 3}+\text{etc.}$$

(24.) Prove that if x_r represent $x(x-1)\ldots(x-r+1)$, and a similar notation apply to y and $x+y$, then

$$(x+y)_r=x_r+rx_{r-1}y_1+\frac{r(r-1)}{1\cdot 2}x_{r-2}y_2+\text{etc.}$$

CHAPTER XIX.

LOGARITHMS AND EXPONENTIAL THEOREM.

249. The logarithm of a number is the exponent of the power to which a second number, called the base, must be raised in order to produce the first. Thus, if $N = a^x$, then x is the logarithm of the number N to the base a. This is usually written $\log_a N = x$.

Suppose 2 to be used as a base, then the logarithm of 8 will be 3 or $\log_2 8 = 3$ for $8 = 2^3$, also $\log_2 64 = 6$ for $64 = 2^6$.

The small figure written to the right of the contraction (*log.*) indicates the base employed.

Since an index may be positive or negative, whole or fractional, a logarithm may also be any of these.

250. From the definition of a logarithm we may easily derive the following THEOREMS :—

I.—*The logarithm of a product is obtained by adding together the logarithms of its factors.*

Let $\log_a N = x$, and $\log_a P = y$.
By definition, $N = a^x$, and $P = a^y$.
By multiplication, $NP = a^x a^y = a^{x+y}$.

$$\text{By definition, } \log_a \mathrm{NP} = x + y.$$
$$\text{But } x = \log_a \mathrm{N} \text{ and } y = \log_a \mathrm{P}.$$
$$\therefore \log_a \mathrm{NP} = \log_a \mathrm{N} + \log_a \mathrm{P}.$$

The same can be proved for any number of factors.

II.—*The logarithm of a quotient is obtained by subtracting the logarithm of the divisor from the logarithm of the dividend.*

$$\text{Let } \log_a \mathrm{N} = x, \text{ and } \log_a \mathrm{P} = y,$$
$$\text{then } \mathrm{N} = a^x, \text{ and } \mathrm{P} = a^y.$$
$$\text{By division, } \frac{\mathrm{N}}{\mathrm{P}} = \frac{a^x}{a^y} = a^{x-y}.$$
$$\text{By definition, } \log_a \frac{\mathrm{N}}{\mathrm{P}} = x - y = \log_a \mathrm{N} - \log_a \mathrm{P}.$$

III.—*The logarithm of the power of a number is obtained by multiplying the logarithm of the number by the index of the power.*

$$\text{Let } \log_a \mathrm{N} = x, \text{ then } \mathrm{N} = a^x.$$

Raise both sides of this equation to the nth power,

$$\mathrm{N}^n = a^{nx}.$$
$$\text{By definition, } \log_a \mathrm{N}^n = nx = n \log_a \mathrm{N}.$$

This must evidently be true whether n be whole or fractional.

IV.—*The logarithms of a number in two different systems are inversely as the logarithms of their bases in any system.*

$$\text{Let } \log_a \mathrm{N} = x, \text{ and } \log_b \mathrm{N} = y,$$
$$\text{then } \mathrm{N} = a^x \text{ and } \mathrm{N} = b^y,$$
$$\therefore a^x = b^y.$$

Take logarithms of these in any base, say e,

$$\text{then } \log_e a^x = \log_e b^y,$$

and by Theorem III., $x \log_e a = y \log_e b$,

$\therefore$ Art. 203, $x : y :: \log_e b : \log_e a$,

that is, $\log_a N : \log_b N :: \log_e b : \log_e a$.

V.—*In every system the logarithm of the base is* 1, *of* 1 *is* 0, *and of* 0 *is* $-\infty$.

Since $a = a^1$, $\log_a a = 1$.

Since $1 = a^0$, $\log_a 1 = 0$.

And since $0 = \frac{1}{a^\infty} = a^{-\infty}$, $\log_a 0 = -\infty$.

Obviously this must hold for any base.

251. Exponential Theorem.—A quantity having a variable exponent is known as an exponential quantity; and the law of its expansion in a series in terms of the exponent is called the Exponential Theorem.

Given the exponential quantity a^y, it is required to expand it in a series in terms of ascending powers of y.

Assume $a^y = m + Ay + By^2 + Cy^3 + Dy^4 + \text{etc.}$ [1]

Square both sides—

$$a^{2y} = m^2 + 2mAy + (2mB + A^2)y^2 + (2mC + 2AB)y^3 + (2mD + 2AC + B^2)y^4 + \text{etc.} \quad [2]$$

Write $2y$ for y in [1], then—

$$a^{2y} = m + 2Ay + 4By^2 + 8Cy^3 + 16Dy^4 + \text{etc.} \quad [3]$$

Equate coefficients of like powers of y (Art. 236)—

Then $m^2 = m$, and $\therefore\ m = 1$

$2A = 2mA$, $\therefore\ A = A$

$4B = 2mB + A^2$, $\therefore\ B = \frac{A^2}{2}$

$8C = 2mC + 2AB$, $\therefore\ C = \frac{A^3}{2 \cdot 3}$

$16D = 2mD + 2AC + B^2$, $\therefore\ D = \frac{A^4}{2 \cdot 3 \cdot 4}$.

. . . . . .

Substitute these values of B, C, etc., in [1],

then $$a^y = 1 + Ay + \frac{A^2y^2}{2} + \frac{A^3y^3}{2.3} + \frac{A^4y^4}{2.3.4} + \text{etc.} \qquad [4]$$

To find the value of A write $a = 1 + \overline{a-1}$, and expand by binomial theorem—

$$a^y = (1+\overline{a-1})^y = 1 + y(a-1) + \frac{y(y-1)}{2}(a-1)^2 + \frac{y(y-1)(y-2)}{2.3}(a-1)^3 + \text{etc.}$$

$$= 1 + \left\{(a-1) - \frac{1}{2}(a-1)^2 + \frac{1}{3}(a-1)^3 - \frac{1}{4}(a-1)^4 + \text{etc.}\right\}y +$$

terms containing higher powers of y. [5]

Equate coefficient of y in [1] with that in [5], then—

$$A = (a-1) - \frac{1}{2}(a-1)^2 + \frac{1}{3}(a-1)^3 - \frac{1}{4}(a-1)^4 + \text{etc.}$$

If in [4] e^x be the value of a^y which results when $Ay = x$, then—

$$e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{2.3} + \frac{x^4}{2.3.4} + \text{etc.} \qquad [6]$$

and if Ay or x be made equal to 1, then—

$$e = 1 + 1 + \frac{1}{2} + \frac{1}{2.3} + \frac{1}{2.3.4} + \text{etc.}$$

$= 2{\cdot}718281828459$ . . . . . by calculating the value of the different terms.

252. Logarithmic Series.—When, as above, $Ay = 1$, we have $a^y = a^{\frac{1}{A}} = e$, and $\therefore\ a = e^A$.

Take logarithms of both sides, and

$$\log_e a = \log_e e^A = A \log_e e \quad \text{(Theorem III.)}$$

$$= A, \text{ since } \log_e e = 1 \quad \text{(Theorem V.)}$$

$$= (a-1) - \frac{1}{2}(a-1)^2 + \frac{1}{3}(a-1)^3 - \frac{1}{4}(a-1)^4 + \text{etc.}$$

Let $a = 1 + x$, then

$$\log_e(1+x) = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4 + \text{etc.} \qquad [7]$$

Instead of x, write $-x$, and the above will become—

$$\log_e(1-x) = -x - \frac{1}{2}x^2 - \frac{1}{3}x^3 - \frac{1}{4}x^4 - \text{etc.} \qquad [8]$$

By subtraction—

$$\log_e(1+x) - \log_e(1-x) = 2\left(x + \frac{1}{3}x^3 + \frac{1}{5}x^5 + \text{etc.}\right)$$

But $\log_e(1+x) - \log_e(1-x) = \log_e\dfrac{1+x}{1-x}$ (Theorem II.)

$$\therefore \log_e\frac{1+x}{1-x} = 2\left(x + \frac{1}{3}x^3 + \frac{1}{5}x^5 + \text{etc.}\right) \qquad [9]$$

Let $x = \dfrac{1}{2n+1}$, then $\dfrac{1+x}{1-x} = \dfrac{n+1}{n}$, and

$$\log_e\left(\frac{n+1}{n}\right) = 2\left\{\frac{1}{2n+1} + \frac{1}{3(2n+1)^3} + \frac{1}{5(2n+1)^5} + \text{etc.}\right\} \qquad [10]$$

$$\text{or } \log_e(n+1) = \log_e n + 2\left\{\frac{1}{2n+1} + \frac{1}{3(2n+1)^3} + \frac{1}{5(2n+1)^5} + \text{etc.}\right\} \qquad [11]$$

253. By means of this formula the logarithm of any number to the base e may be calculated, for when $\log_e n$ is known, $\log_e(n+1)$ may be found.

The following exhibits this calculation for the first ten natural numbers :—

$\log_e 1 = 0$ (Theorem V.)

$$\log_e 2 = \log_e 1 + 2\left(\frac{1}{3} + \frac{1}{3 \cdot 3^3} + \frac{1}{5 \cdot 3^5} + \frac{1}{7 \cdot 3^7} + \text{etc.}\right)$$

$$= \cdot 6931472$$

$$\log_e 3 = \log_e 2 + 2\left(\frac{1}{5} + \frac{1}{3 \cdot 5^3} + \frac{1}{5 \cdot 5^5} + \frac{1}{7 \cdot 5^7} + \text{etc.}\right)$$

$$= 1 \cdot 0986122$$

$$\log_e 4 = \log_e 2^2 = 2\log_e 2 = \quad 1{\cdot}3862944$$

$$\log_e 5 = \log 4 + 2\left(\frac{1}{9} + \frac{1}{3\,.\,9^3} + \frac{1}{5\,.\,9^5} + \frac{1}{7\,.\,9^7} + \text{etc.}\right)$$

$$= \quad 1{\cdot}6094379$$

$$\log_e 6 = \log_e 2 + \log_e 3 = \quad 1{\cdot}7917595$$

$$\log_e 7 = \log 6 + 2\left(\frac{1}{13} + \frac{1}{3\,.\,13^3} + \frac{1}{5\,.\,13^5} + \text{etc.}\right)$$

$$= \quad 1{\cdot}9459101$$

$$\log_e 8 = \log_e 2^3 = 3\log_e 2 = \quad 2{\cdot}0794415$$

$$\log_e 9 = \log_e 3^2 = 2\log_e 3 = \quad 2{\cdot}1972246$$

$$\log_e 10 = \log_e 2 + \log_e 5 = \quad 2{\cdot}3025851$$

In obtaining logarithms for higher numbers, the work can in many cases be shortened by the employment of arithmetical artifices.

Although any number may be employed as a base, only two systems of logarithms are tabulated—namely, that in which the base is e, or $2{\cdot}718281828459\ldots.$ and that in which the base is 10. Logarithms in which the base is e are called natural or Napierian logarithms, and are employed in mathematical analysis; those in which the base is 10 are known as common, or Briggs's logarithms, and are much used in arithmetical and trigonometrical calculations.

254. From Theorem IV., by changing b into e, we have

$$\log_a N : \log_e N :: \log_e e : \log_e a,$$

and since $\log_e e = 1$

$$\log_a N = \frac{1}{\log_e a} \log_e N.$$

Whereby it appears that the logarithm of a number to any base may be found by multiplying the Napierian logarithm of the number by the reciprocal of the Napierian logarithm of the new base.

This multiplier is usually called the Modulus, and represented by M.

255. Common Logarithms.—If in the above expression we put $a = 10$, then

$$\log_{10} N = \frac{1}{\log_e 10} \log_e N = M \log_e N,$$

or, representing $\log_{10}$ by Log, and $\log_e$ by log,

$$\text{Log } N = M \log N.$$

That is, the common logarithm of a number may be found by multiplying the Napierian logarithm of the number by the modulus.

Substituting $\frac{n+1}{n}$ for N, and using the value of $\log\frac{n+1}{n}$ obtained in [10], we have—

$$\text{Log}\frac{n+1}{n} = 2M \log\frac{n+1}{n}$$

$$= 2M \left\{ \frac{1}{2n+1} + \frac{1}{3(2n+1)^3} + \frac{1}{5(2n+1)^5} + \text{etc.} \right\}$$

or $\text{Log}(n+1)$

$$= \text{Log } n + 2M \left\{ \frac{1}{2n+1} + \frac{1}{3(2n+1)^3} + \frac{1}{5(2n+1)^5} + \text{etc.} \right\} \quad [12]$$

From this formula, the common logarithm of any number may be calculated when that of the previous number is known.

Let it be required to find Log 11.

$$M = \frac{1}{\log_e 10} = \frac{1}{2{\cdot}302585092994} = {\cdot}434294481903 + \text{etc.}$$

$$\therefore \text{Log } 11 = \text{Log } 10 + {\cdot}86858896 \left(\frac{1}{21} + \frac{1}{3.21^3} + \frac{1}{5.21^5} + \text{etc.} \right)$$

$$= 1{\cdot}04139268.$$

Note.—The number of figures used in the modulus may be lessened or increased according to the degree of accuracy required in the logarithm.

256. The integral part of a logarithm is called the *characteristic*, and the decimal part the *mantissa*.

In the system of common logarithms, since $1=10^0$, $10=10^1$, $100=10^2$, $1000=10^3$, etc., then $\text{Log}\,1=0$, $\text{Log}\,10=1$, $\text{Log}\,100=2$, $\text{Log}\,1000=3$, etc., and for a number between

1 and 10, the Log is between 0 and 1, with 0 for characteristic.
10 and 100, the Log is between 1 and 2, with 1 for characteristic.
100 and 1000, the Log is between 2 and 3, with 2 for characteristic.
etc. etc. etc. etc.
10^n and 10^{n+1}, the Log is between n and $n+1$, with n for characteristic.

So that the characteristic is always one less than the number of integral places the number contains.

Thus the characteristic of $\text{Log}\,251$ is 2, and that of $\text{Log}\,3456{\cdot}75$ is 3.

$$\text{Since } \frac{1}{10}=10^{-1},\ \frac{1}{100}=10^{-2},\ \text{etc.},$$

$$\text{then } \text{Log}\frac{1}{10}=-1,\ \text{Log}\frac{1}{100}=-2,\ \text{etc.}$$

From which the logarithm of a number between 1 and $\frac{1}{10}$ must be less than 0, and greater than -1; that is, it is a negative fraction. It may also be represented by $-1+$ a fraction, and this method is generally adopted.

Therefore, for a number between

1 and ·1, the Log is between 0 and −1, with −1 for characteristic.
·1 and ·01, the Log is between −1 and −2, with −2 for characteristic.
·01 and ·001, the Log is between −2 and −3, with −3 for characteristic.
etc. etc. etc. etc.
$(\cdot1)^n$ and $(\cdot1)^{n+1}$, the Log is between $-n$ and $-(n+1)$, with $-(n+1)$ for „

where n is the number of ciphers following the point.

The characteristic of Log·0425 is -2, and that of Log·000025 is -5.

257. In the tables of common logarithms, opposite the number 5432, in the logarithmic column, will be found 7349598. This, however, is only the mantissa of the logarithm, the characteristic having to be supplied. By the rule given above, it is found to be 3, and therefore Log 5432 = 3·7349598.

Let it now be required to find Log 543·2.

$$\text{Log } 543{\cdot}2 = \text{Log}\frac{5432}{10} = \text{Log } 5432 - \text{Log } 10$$
$$= 3{\cdot}7349598 - 1 = 2{\cdot}7349598,$$

the mantissa being the same as before.

$$\text{Also Log } 5{\cdot}432 = \text{Log}\frac{5432}{1000} = \text{Log } 5432 - \text{Log } 1000$$
$$= 3{\cdot}7349598 - 3 = {\cdot}7349598,$$

the mantissa still remaining unchanged.

$$\text{Also Log}{\cdot}5432 = \text{Log}\frac{5432}{10000} = \text{Log } 5432 - \text{Log } 10000$$
$$= 3{\cdot}7349598 - 4 = {\cdot}7349598 - 1$$
$$= \bar{1}{\cdot}7349598,$$

the characteristic being written with its sign above it, as it only, and not the mantissa, is negative.

$$\text{So also Log}{\cdot}005432 = \text{Log}\frac{5432}{10^6} = \text{Log } 5432 - \log 10^6$$
$$= 3{\cdot}7349598 - 6 = \bar{3}{\cdot}7349598.$$

From all this, it appears that the mantissa is invariable for the logarithms of all numbers consisting of the same successive digits, and that the characteristic is determined by the position of the decimal point.

The operations of addition, subtraction, multiplication,

and division are performed on positive logarithms as on ordinary numbers; those with negative characteristics follow the rules of algebra.

Illustrative Examples.

(1.) Find the base used when $\log 32 = 2{\cdot}5$.

Let x = the base,

then by definition, $x^{2\cdot5} = 32 = 2^5$.

$$\therefore\ x = 2^{\frac{5}{2\cdot5}} = 2^2 = 4.$$

(2.) Given $\text{Log}\,2 = {\cdot}3010300$, and $\text{Log}\,3 = {\cdot}4771213$, to find $\text{Log}\,648$ and $\text{Log}\,{\cdot}0045$.

$$\text{Log}\,648 = \text{Log}(8 \times 81) = \text{Log}\,2^3 + \text{Log}\,3^4 = 3\,\text{Log}\,2 + 4\,\text{Log}\,3$$
$$= {\cdot}9030900 + 1{\cdot}9084852 = 2{\cdot}8115752.$$

$$\text{Log}\,{\cdot}0045 = \text{Log}\,\frac{45}{10000} = \text{Log}\,\frac{9}{2000} = \text{Log}\,9 - \text{Log}\,2000$$
$$= 2\,\text{Log}\,3 - \text{Log}\,2 - \text{Log}\,1000$$
$$= {\cdot}9542426 - {\cdot}3010300 - 3$$
$$= \bar{3}{\cdot}6532126.$$

(3.) From the Formula [12] in Art. 255 calculate the common logarithm of 26, $\text{Log}\,2 = {\cdot}3010300$ being given.

$$\text{Log}\,26 = \text{Log}\,25 + 2 \times {\cdot}4342944\left(\frac{1}{51} + \frac{1}{3\,.\,51^3} + \text{etc.}\right)$$

$$\text{Log}\,25 = \text{Log}\,\frac{100}{4} = \text{Log}\,100 - \text{Log}\,4 = 2 - 2\,\text{Log}\,2$$
$$= 2 - {\cdot}6020600 = \qquad 1{\cdot}3979400$$
$${\cdot}8685889\left(\frac{1}{51} + \frac{1}{3\,.\,51^3}\right) = \quad {\cdot}0170333$$
$$\therefore\ \text{Log}\,26 = \quad 1{\cdot}4149733$$

(4.) Find the value of x and y from the equations $a^x = b^y$ and $ax^m = by^n$.

Extract nth root of second equation—

$$a^{\frac{1}{n}}x^{\frac{m}{n}} = b^{\frac{1}{n}}y$$

$$\therefore\ y = \left(\frac{a}{b}\right)^{\frac{1}{n}}x^{\frac{m}{n}}$$

Take logarithms of first equation, then, by Theorem III.,

$$x \log a = y \log b.$$

Substitute in this the value of y obtained above—

$$x \log a = \left(\frac{a}{b}\right)^{\frac{1}{n}}x^{\frac{m}{n}} \log b.$$

Divide by $x^{\frac{m}{n}} \log a$—

$$x^{\frac{n-m}{n}} = \left(\frac{a}{b}\right)^{\frac{1}{n}} \frac{\log b}{\log a}$$

$$\therefore\ x = \left\{ \frac{a}{b}\left(\frac{\log b}{\log a}\right)^{n} \right\}^{\frac{1}{n-m}}$$

Similarly, or by substitution, we find—

$$y = \left\{ \frac{a}{b}\left(\frac{\log b}{\log a}\right)^{m} \right\}^{\frac{1}{n-m}}$$

(5.) Bring ·0625 to the power denoted by $\frac{8}{9}$, having given Log 2 = ·3010300, and Log 8504937 = 6·9296711.

$$\begin{aligned}\text{Log } \cdot 0625 &= \text{Log } \frac{625}{10000} = \text{Log } \frac{1}{16} = \text{Log } 1 - \text{Log } 16 \\ &= 0 - 4 \text{Log } 2 = -1 \cdot 2041200 \\ &= \bar{2} \cdot 7958800.\end{aligned}$$

The whole logarithm being negative, the *mantissa* is subtracted from 1 in order to render it positive.

By Theorem III., $\text{Log } (\cdot 0625)^{\frac{8}{9}} = \frac{8}{9} \text{Log } \cdot 0625$.

$$\text{Log } \cdot 0625 = \bar{2} \cdot 7958800$$

$$9\overline{)\ \bar{1}0 \cdot 3670400}\ \ 8$$

$$\bar{2} \cdot 9296711$$

In multiplying, there is a carriage of $+6$ from the mantissa, and as the characteristic becomes -16, these added together make -10, which is written $\overline{10}$. In dividing, $\overline{10}$ is changed into $\overline{18}$, so as to be exactly divisible by 9, and 8 is prefixed to the next figure, making 8·3 to be divided by 9.

The work may be arranged thus—

$$\frac{-18+8\cdot3670400}{9} = -2+\cdot92+\text{etc.} = \bar{2}\cdot92+\text{etc.}$$

Since the mantissa of the answer is the same as that of the logarithm of the given number 8504937, and the characteristic is $\bar{2}$, the above is the logarithm of ·08504937, and therefore

$$(\cdot0625)^{\frac{8}{9}} = \cdot08504937.$$

(6.) Prove that

$$\log_e x = \left(x-\frac{1}{x}\right) - \frac{1}{2}\left(x^2-\frac{1}{x^2}\right) + \frac{1}{3}\left(x^3-\frac{1}{x^3}\right) - \text{etc.}$$

By [6], Art. 252, $\log_e(1+x) = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \text{etc.}$

For x put $\frac{1}{x}$, $\log_e\left(1+\frac{1}{x}\right) = \frac{1}{x} - \frac{1}{2}\cdot\frac{1}{x^2} + \frac{1}{3}\cdot\frac{1}{x^3} - \text{etc.}$

Subtract under from upper line—

$$\log_e(1+x) - \log_e\left(1+\frac{1}{x}\right) = \left(x-\frac{1}{x}\right) - \frac{1}{2}\left(x^2-\frac{1}{x^2}\right) + \frac{1}{3}\left(x^3-\frac{1}{x^3}\right) - \text{etc.}$$

But $\log_e(1+x) - \log_e\left(1+\frac{1}{x}\right) = \log_e \dfrac{1+x}{1+\frac{1}{x}}$, (Theorem II.)

$$= \log_e \frac{(1+x)x}{x+1} = \log_e x.$$

$$\therefore\ \log_e x = \left(x-\frac{1}{x}\right) - \frac{1}{2}\left(x^2-\frac{1}{x^2}\right) + \frac{1}{3}\left(x^3-\frac{1}{x^3}\right) - \text{etc.}$$

EXAMPLES FOR PRACTICE.—LXXX.

(1.) From the definitions, find the base used when $\log 16 = 4$, when $\log 6\frac{1}{4} = 2$, and when $\log 4 = \frac{2}{3}$.

(2.) Find the logarithm of 1296 to the base 6, of 6·25 to the base 2·5, and of 1728 to the base $2\sqrt{3}$.

(3.) What is the characteristic of 9 to the base 2? What of 15 to the base 4, and of 1000 to the base 5?

(4.) The logarithms of a, b, and c being known, show how to find $\log \frac{a^3b^2}{c}$, and $\log \frac{\sqrt[3]{a^2b}}{\sqrt[4]{bc^3}}$.

(5.) If $\log 75 = a$ and $\log 45 = b$, what is the logarithm of 15?

(6.) Given $\log 3 = m$, and $\log 5 = n$, to find the logarithms of 15, 135, 375, $\frac{3}{5}$, and $16\frac{1}{5}$.

(7.) The common logarithm of 2 being ·3010300, find the common logarithm of 5, of 25, and of ·125.

(8.) Given the common logarithm of 2 as above, and that of 3 = ·4771213, to find $\log 48$, $\log 225$, and $\log$ ·006.

(9.) The common logarithms of 2 and 3 being as in previous example, find those of $\left(\frac{5}{8}\right)^3$, $\left(\frac{4}{15}\right)^{\frac{1}{3}}$, and $\left(\frac{2}{9}\right)^{\frac{2}{3}}$.

(10.) Given the Napierian logarithms of 1, 2, 3, ... 10, as in Art. 253, to find those of 13 and 31.

(11.) From the Napierian logarithms of 13 and 31 found in last example, derive the common logarithms of the same numbers.

(12.) From the formula [12] in Art. 255, calculate the common logarithms of 101 and 257.

(13.) Making use of such of the logarithms given or found in previous examples as may be necessary, say how many figures will be required to express the twentieth power of 12, and the tenth power of 31.

(14.) Given Log 392 = 2·5932860, and Log 448 = 2·6512780, to find Log 686, and Log ·0875.

(15.) Find the value of x in each of the exponential equations, $20^x = 100$, and $16^x \times 9^x = 125$.

(16.) Solve the equations,

(I.) $\log x - \log y = \log a - \log b$, $x^2 - y^2 = a + b$.

(II.) $a \log x = b \log y$, $x \log a = y \log b$.

(17.) Show that

$$\log_e y^2 = \log_e (y^2 - 1) + 2\left\{ \frac{1}{2y^2 - 1} + \frac{1}{3(2y^2 - 1)^3} + \text{etc.} \right\}$$

(18.) If $\frac{1}{a}$, $\frac{1}{b}$ are the roots of the equation $x^2 - px + q = 0$, prove that

$$\log_e (x^2 + px + q) = \log_e q + (a+b)x - \frac{1}{2}(a^2+b^2)x^2 + \frac{1}{3}(a^3+b^3)x^3 - \text{etc.}$$

CHAPTER XX.

INTEREST AND ANNUITIES.

258. Interest.—The terms used in interest are so well known as to render it unnecessary to give an explanation of them here.

We shall adopt the following abbreviations :—

Let P = the principal or money lent.
r = the rate per cent. per annum.
n = the number of years.
I = the interest.
M = the amount.
R = the amount of *one pound* for *one year*.

259. Simple Interest.—If r be the rate per cent., or interest on £100 for one year, the interest on one pound for a year will be $\frac{r}{100}$, and on P pounds for a year $\frac{Pr}{100}$.

If the interest on a sum of money for one year be $\frac{Pr}{100}$, that on the same sum for n years will evidently be n times $\frac{Pr}{100}$; or $\frac{Pnr}{100} = I$.

Plainly also $M = P + I = P + \frac{Pnr}{100} = P\left(1 + \frac{nr}{100}\right)$.

From these equations, when any three of the quantities are given, the other two can be found.

260. Compound Interest.—If $\frac{Pr}{100}$ be the interest on a sum of P pounds for one year, the amount at the end of the year will be

$$P + \frac{Pr}{100} = P\left(1 + \frac{r}{100}\right) = PR. \quad \text{(Art. 258.)}$$

The interest on this sum for one year will be $\frac{PRr}{100}$, and its amount at the end of the year will be

$$PR + \frac{PRr}{100} = PR\left(1 + \frac{r}{100}\right) = PR^2.$$

Similarly the interest on this will be $\frac{PR^2r}{100}$, and its amount

$$PR^2 + \frac{PR^2r}{100} = PR^2\left(1 + \frac{r}{100}\right) = PR^3.$$

So that when Compound Interest is reckoned—

The amount of P pounds for 1 year is PR;
The amount of P pounds for 2 years is PR^2;
The amount of P pounds for 3 years is PR^3;

and so on for any number of years.

We have $\therefore M = PR^n$

and $I = M - P = PR^n - P = P(R^n - 1)$.

Note.—This result is strictly accurate only when n is an exact number of years. If $n = m + \frac{p}{q}$, where $\frac{p}{q}$ is a proper fraction, then

$$M = PR^m\left(1 + \frac{p}{q} \cdot \frac{r}{100}\right).$$

261. Interest is generally payable annually, but it may be paid quarterly, monthly, etc., in which case, while the payments are more numerous, the sum added each time will be less.

Suppose the payments made q times a year, then the interest per pound on each occasion will be $\frac{r}{100} \times \frac{1}{q}$, and the number of payments will be qn.

As in last article, it can be shown that

$$M = P\left(1 + \frac{1}{q} \cdot \frac{r}{100}\right)^{qn}.$$

262. From this we may find the amount of a sum of money laid out at compound interest for a number of years, the interest being due every instant.

Expand $\left(1 + \frac{r}{100q}\right)^{qn}$ by the Binomial Theorem, then

$$M = P\left\{1 + qn \cdot \frac{r}{100q} + \frac{qn(qn-1)}{1 \cdot 2}\left(\frac{r}{100q}\right)^2 + \frac{qn(qn-1)(qn-2)}{1 \cdot 2 \cdot 3}\left(\frac{r}{100q}\right)^3 + \text{etc.}\right\}$$

$$= P\left\{1 + \frac{nr}{100} + \frac{n\left(n-\frac{1}{q}\right)}{1 \cdot 2}\left(\frac{r}{100}\right)^2 + \frac{n\left(n-\frac{1}{q}\right)\left(n-\frac{2}{q}\right)}{1 \cdot 2 \cdot 3}\left(\frac{r}{100}\right)^3 + \text{etc.}\right\}$$

and since q is very great, $\frac{1}{q}$, $\frac{2}{q}$, etc., are very small, and may be put $= 0$.

$$\therefore M = P\left\{1 + \frac{nr}{100} + \frac{1}{1 \cdot 2}\left(\frac{nr}{100}\right)^2 + \frac{1}{1 \cdot 2 \cdot 3}\left(\frac{nr}{100}\right)^3 + \text{etc.}\right\}$$

$$= Pe^{\frac{nr}{100}}, \text{ by [6], Art. 251.}$$

263. Present Worth.—The sum P which on the lapse of a certain time, by the accumulation of interest, becomes M, is the present worth of M, and since $PR^n = M$, we have $P = \frac{M}{R^n}$.

264. Annuities.—If an annuity of A pounds, payable at the end of each year, be left unpaid for a number of years, it will amount

At the end of the first year to A;
At the end of the second year to $AR + A$;
At the end of the third year to $AR^2 + AR + A$.
. . . . . .
At the end of the nth year to $A(R^{n-1} + R^{n-2} + \ldots + R + 1)$,

or $A\left(\frac{R^n - 1}{R - 1}\right)$. Art. 214.

265. If P be the present value of the annuity, we have, by Art. 263—

$$P = \frac{A(R^n - 1)}{R - 1} \div R^n.$$
$$= \frac{A(R^n - 1)}{R^n(R - 1)}.$$

266. If n be supposed very large, then $\frac{1}{R^n}$ may be put $= 0$, and we shall have—

$$P = \frac{A(R^n - 1)}{R^n(R - 1)} = A\frac{\left(1 - \frac{1}{R^n}\right)}{R - 1}$$
$$= \frac{A}{R - 1} = \frac{100A}{r},$$

the present value of an annuity in perpetuity.

267. When an annuity is to commence at the end of p years, and to continue thereafter for q years, its present value will be the present value for $p+q$ years, minus that for p years.

$$\text{Or } P = \frac{A(R^{p+q}-1)}{R^{p+q}(R-1)} - \frac{A(R^{p}-1)}{R^{p}(R-1)} = \frac{A(R^{q}-1)}{R^{p+q}(R-1)}.$$

268. Also, since

$$\frac{A(R^{q}-1)}{R^{p+q}(R-1)} = \frac{A}{R^{p}(R-1)} - \frac{A}{R^{p+q}(R-1)}$$

by making q very large, we have—

$$P = \frac{A}{R^{p}(R-1)} = \frac{100A}{rR^{p}},$$

the present value of a deferred perpetual annuity.

Illustrative Examples.

(1.) Find the amount of £500, for 10 years, at 5 per cent. compound interest.

By the formula, Art. 260,

$$M = PR^{n} = 500\left(1+\frac{5}{100}\right)^{10}$$
$$= 500(1{\cdot}05)^{10}.$$

The solution may be completed by the ordinary processes of arithmetic, the answer being—

$$500 \times 1{\cdot}628894 = £814{\cdot}447;$$

or by logarithms, in which case we have—

$$\text{Log } M = \text{Log } 500 + 10 \text{ Log } 1{\cdot}05.$$

From a table of logarithms we find Log 500 and Log 1·05, and set them down as under:—

$$\begin{array}{r} \text{Log } 1{\cdot}05 = 0{\cdot}0211893 \\ \hline 10 \text{ Log } 1{\cdot}05 = 0{\cdot}2118930 \\ \text{Log } 500 = 2{\cdot}6989700 \\ \hline 2{\cdot}9108630 \end{array}$$

By another reference to the tables, we find that this sum is the logarithm of 814·447; so that the answer is the same by both methods.

(2.) In what time will a sum of money double itself at 4 per cent. compound interest?

By the question we have—

$$PR^n = 2P,$$

$$\text{that is,} \quad R^n = (1{\cdot}04)^n = 2.$$

Taking logarithms of both sides—

$$\text{then,} \quad n \text{ Log } 1{\cdot}04 = \text{Log } 2,$$

$$\text{and} \quad n = \frac{\text{Log } 2}{\text{Log } 1{\cdot}04} = \frac{{\cdot}3010300}{{\cdot}0170333} = 17{\cdot}67.$$

Log 1·04 may be calculated from some of those given or found in previous examples, for

$$\text{Log } 1{\cdot}04 = \text{Log } \frac{104}{100} = \text{Log } 104 - \text{Log } 100 = \text{Log } 13 + 3 \log 2 - 2.$$

(3.) Find the present value of an annuity of £30, to continue for 40 years, interest being at the rate of 6 per cent.

$$\text{By the formula, } P = \frac{A(R^n - 1)}{R^n(R - 1)} = \frac{100A(1 - R^{-n})}{r}.$$

$$\text{First find } \text{Log } R^{-n} = -n \text{ Log } R = -40 \text{ Log } 1{\cdot}06.$$

From the tables, Log 1·06 = ·0253059,

$$\therefore -40\,\text{Log}\,1{\cdot}06 = -1{\cdot}0122360 = \bar{2}{\cdot}9877640 = \text{Log}\cdot0972219.$$

$$\therefore\ P = \frac{100 \times 30(1 - {\cdot}0972219)}{6}$$

$$= 500 \times {\cdot}9027781 = 451{\cdot}38905$$

$$= £451,\ 7s.\ 9\tfrac{1}{2}d.$$

(4.) Four persons, H, I, K, L, contribute equal sums towards the purchase of an estate. How long may the first three enjoy it in succession, so that the fourth may be entitled to the absolute reversion?

Let x, y, z = the required times in order, and h, i, k, l = successively the present values of the different occupancies.

Then, by Art. 265, $h = \dfrac{A}{R-1} \cdot \dfrac{R^x - 1}{R^x}$; [1]

Then, by Art. 267, $i = \dfrac{A}{R-1} \cdot \dfrac{R^y - 1}{R^{x+y}}$; [2]

Then, by Art. 267, $k = \dfrac{A}{R-1} \cdot \dfrac{R^z - 1}{R^{x+y+z}}$; [3]

Then, by Art. 268, $l = \dfrac{A}{R-1} \cdot \dfrac{1}{R^{x+y+z}}$. [4]

As these values are all equal, we have—

From [3] and [4], $R^z = 2$, and $z = \dfrac{\text{Log}\,2}{\text{Log}\,R}$.

From [2] and [3], $R^y = \dfrac{3}{2}$, and $y = \dfrac{\text{Log}\,3 - \text{Log}\,2}{\text{Log}\,R}$.

From [1] and [2], $R^x = \dfrac{4}{3}$, and $x = \dfrac{\text{Log}\,4 - \text{Log}\,3}{\text{Log}\,R}$.

If interest be taken at 5 per cent., it will be found that the successive times are nearly 6, 8, and 14 years.

EXAMPLES FOR PRACTICE.—LXXXI.

(1.) Find the difference between the simple and the compound interest on £100 for 3 years at 3½ per cent.

(2.) What is the amount of £600 laid out at 5 per cent. compound interest for 16 years?

Given Log 6 = ·7781513
Given Log 1·05 = ·0211893
Given Log 1·309725 = ·1171801.

(3.) Find the present worth of £800, due 10 years hence, reckoning compound interest at 4 per cent.

Given Log 2 = ·3010300
Given Log 1·04 = ·0170333
Given Log 5·40452 = ·7327570.

(4.) How long would it take for £1 to become £10, at 6 per cent. compound interest?

Given Log 1·06 = ·0253059.

(5.) What will £10,000 amount to in 5 years, at 5 per cent. per annum, the interest being added quarterly?

Given Log 1·0125 = ·0053950
and Log 1·282 = ·1079000.

(6.) What would be the amount of £1000 in 10 years, at 5 per cent., if the interest were payable every instant? Also, in what time would £1000 become £2000 on the same conditions?

(7.) The heir of an estate being a minor, the rents accumulated at the rate of £3250 a year for 8 years, with interest at 4 per cent.: what was their amount when he became of age?

Given Log 1·04; and Log 1·368568 = ·1362664.

(8.) In five years B will be entitled to an annuity of

£40, to last for 25 years. If interest be reckoned at 4 per cent., what is the present value of the annuity?

Given Log 1·04; and Log 2·66583 = ·4258325,
and Log 3·243388 = ·5109990.

(9.) Ten years' purchase is paid for an annuity to continue a certain time, and 16 years' purchase for another which is to continue twice as long. At what rate is interest reckoned?

(10.) A property worth £1000 is to be paid for in yearly instalments of £100. In what time will it be free of debt, the first instalment being due at the end of the year, and interest being allowed at 5 per cent.?

(11.) Three persons buy an estate jointly, the first paying one-half of the price, the second one-third of it, and the third the remainder. How long should the first two enjoy the estate in succession, so that it may afterwards become the absolute property of the third?

(12.) A farm is offered on a lease of 20 years at a rent of £150, or of £135 with a present payment of £200. Interest being at 5 per cent., which offer is the more advantageous to the tenant?

Given Log 1·05, and Log 2·653298 = ·4237860.

MISCELLANEOUS EXAMPLES.

Principally selected from other works.

(1.) Find the square root of

$$\frac{x^2}{y^2}+\frac{y^2}{x^2}-2\left(\frac{x}{y}+\frac{y}{x}\right)+3.$$

(2.) Required the cube root of

$$x^6-6x^5+15x^4-20x^3+15x^2-6x+1.$$

(3.) Simplify $\left(\frac{ax}{y}\right)^{\frac{1}{2}}\times\left(\frac{by}{x^2}\right)^{\frac{1}{3}}\times\left(\frac{x^2}{a^3b^2}\right)^{\frac{1}{6}}$.

(4.) Solve by inspection the equations—

(I.) $x^2=18x-77$.

(II.) $x^2-16x=36$.

(5.) Find the values of the unknown quantity in the following:—

$$\left(\frac{x-2}{x+2}\right)^{\frac{1}{2}}+\left(\frac{x+2}{x-2}\right)^{\frac{1}{2}}=4.$$

(6.) Extract the square root of the following:—

$$4x^2(x^2-y)+y^3(y-2)+y^2(4x^2+1).$$

(7.) Simplify $\left(\frac{x^{p+q}}{x^q}\right)^p\div\left(\frac{x^q}{x^{q-p}}\right)^{p-q}$.

(8.) Divide $3a+6a^{\frac{7}{8}}-a^{\frac{5}{4}}+2a^{\frac{4}{3}}-2a^{1\frac{7}{12}}+4a^{\frac{3}{2}}$ by $3a^{\frac{1}{2}}-a^{\frac{3}{4}}+2a^{\frac{5}{8}}$.

(9.) A person bought a certain number of oxen for £240, and, after losing 3, sold the rest for £8 a head more than they cost him, thus gaining £59 by the bargain. What number did he buy?

(10.) Approximate to the ratio of $501^{\frac{1}{2}}:500^{\frac{1}{2}}$, and of $501^{\frac{1}{3}}:500^{\frac{1}{3}}$.

(11.) By performing the operation for extracting the square root, find a value of x which will make $x^4+6x^3+11x^2+3x+31$ a perfect square.

(12.) Solve in positive integers, $39x-56y=11$.

(13.) Extract the cube root of $x^6 + 6x^5 - 40x^3 + 96x - 64$.

(14.) Show that $\dfrac{(3+\sqrt{3})(3+\sqrt{5})(\sqrt{5}-2)}{(5-\sqrt{5})(1+\sqrt{3})} = \sqrt{\dfrac{3}{5}}$.

(15.) Find the sum of the arithmetical progression $\dfrac{1}{n} + \dfrac{n+1}{2n} + \ldots$ to n terms.

(16.) Find the square root of $1 - 2\sqrt{x} + 3x - 2x\sqrt{x} + x^2$.

(17.) If $a : b = b : c$,

then $a+b+c : a-b+c :: (a+b+c)^2 : a^2+b^2+c^2$.

(18.) Solve the equation—

$$\sqrt{(x-1)(2-x)} = \sqrt{3-x^2} + \sqrt{3x-5}.$$

(19.) Simplify $\dfrac{xy}{y-m} \pm \left\{ \dfrac{x^2y^2}{(y-m)^2} - \dfrac{x^2y}{y-m} \right\}^{\frac{1}{2}}$.

(20.) Find the square root of $5 - 12\sqrt{-1}$, and the value of $\sqrt{4+6\sqrt{-5}} + \sqrt{4-6\sqrt{-5}}$.

(21.) A farmer bought oxen, sheep, and hens. The whole number bought was 100, and the whole price £100. If the oxen cost £5, the sheep £1, and the hens 1s. each, how many of each had he?

(22.) If $a^{m^n} = (a^m)^n$, find m in terms of n.

(23.) Solve by inspection the following equations:—

(I.) $110x^2 - 21x + 1 = 0$.

(II.) $x^2 + 4x = 320$.

(24.) What are the values of x and y, when

$$x : 27 :: y : 9 :: 2 : x - y?$$

(25.) Insert 5 arithmetical means between $-\frac{1}{6}$ and $\frac{1}{6}$.

(26.) Reduce to their simplest form

$$\frac{(1+\sqrt{2})^2 + \sqrt{3}(1+\sqrt{2})}{1+\sqrt{2}+\sqrt{3}} \text{ and } \sqrt{2\sqrt{2}}\sqrt{\frac{1+\sqrt{2}+\sqrt{3}}{1+\sqrt{2}-\sqrt{3}}}.$$

(27.) Find the sum of the geometrical series

$$\sqrt{\tfrac{5}{2}} + \sqrt{\tfrac{1}{2}} + \ldots \text{ to infinity.}$$

(28.) Solve the equations—

$$xy = (x - \tfrac{3}{7})(y + \tfrac{2}{3}).$$
$$x^2y^2 = (x^2 + 3)(y^2 - 4).$$

(29.) Three chickens and one duck sold for as much as two geese; and one chicken, two ducks, and three geese were sold together for 25s. What was the price of each?

(30.) Extract the square root of $6 + \sqrt{-13}$.

(31.) Find the number of peals which can be rung on eight bells, the great bell always coming last.

(32.) In a mile race between a bicycle and a tricycle, their rates were proportional to 5 and 4. The tricycle had half a minute's start, but was beaten by 176 yards. Find the rates of each.

(33.) If $A \propto B$ and $B^3 \propto C^2$, express how A varies in respect of C.

(34.) Find what c and d must be, in order that the quantity $a^3x^3 + bx^2 + cx + d$ may be a complete cube.

(35.) Given $\left(x^2 - \frac{a^4}{x^2}\right)^{\frac{1}{2}} + \left(a^2 - \frac{a^4}{x^2}\right)^{\frac{1}{2}} = \frac{x^2}{a}$, to find x.

(36.) One hundred stones are placed at the distance of a yard from each other, in a right line with a basket which is distant one yard from that next to it. A person starts from the basket, and brings them one by one into it. What space does he travel over?

(37.) If $x \propto \frac{z}{y^2}$ and $z^2 \propto \frac{y}{x}$, show that $x \propto \frac{1}{y} \propto \frac{1}{z}$.

(38.) Simplify $\left(\sqrt[4]{\frac{a}{b}} + \sqrt[4]{\frac{b}{a}}\right)$ and $(a+b)\frac{\sqrt{\frac{a}{b}} - \sqrt{\frac{b}{a}}}{\sqrt{\frac{a}{b}} + \sqrt{\frac{b}{a}}}$.

(39.) A basket contains 12 apples and 8 pears. Three apples and two pears are offered for a penny. In how many different ways may a boy select his pennyworth?

(40.) A woman, counting her eggs, reports that when

she counts by threes there are two over, and when she counts by sixes there are three over. Is she right?

(41.) Solve the equation $\frac{(1+x)^5}{1+x^5}=5$.

(42.) Show that the ratio $a:b$ is the duplicate of the ratio $a+c:b+c$, if $c^2=ab$.

(43.) In an arithmetical progression, if $s=pn+qn^2$ whatever be the value of n, find the mth term.

(44.) Solve the equations,

$$\frac{1}{x}+\frac{1}{y}=\frac{1}{2},$$
$$x^2y+xy^2=162.$$

(45.) Find the value of

$$\frac{ax+1+\sqrt{(a^2x^2-1)}}{ax+1-\sqrt{(a^2x^2-1)}} \text{ when } x=\frac{2}{\sqrt{4ab-b^2}}.$$

(46.) The time which an express train takes to travel a journey of 180 miles is to that taken by an ordinary train as 9 : 14. The ordinary train loses as much time from stoppages as it would take to travel 30 miles without stopping. The express train only loses half as much time as the other in this manner, and it also travels 15 miles an hour quicker. Supposing the rates of travelling uniform, what are they in miles per hour?

(47.) Simplify $\frac{\frac{1}{8}\{\sqrt{5}\mp 1\}}{\frac{1}{4\sqrt{2}}\sqrt{(5\pm\sqrt{5})}}$.

(48.) Solve the equation, $x^4-2x^3+x=132$.

(49.) Prove that if $\frac{a_1+a_2x}{a_2+a_3y}=\frac{a_2+a_3x}{a_3+a_1y}=\frac{a_3+a_1x}{a_1+a_2y}$, each of these ratios is equal to $\frac{1+x}{1+y}$, supposing $a_1+a_2+a_3$ not to be zero.

(50.) Divide £500 among A, B, and C, in the proportion of $\frac{1}{3}$, $\frac{1}{4}$, $\frac{1}{5}$; and if A's portion be to B's :: 9 : 8, and to C's :: 6 : 5, show that the shares of A, B, and C are in the proportion of $1\frac{1}{2}$, $1\frac{1}{3}$, $1\frac{1}{4}$.

(51.) Sum $2.5 - 3.9 + 4.13 - 5.17 + 6.21 - 7.25$, etc., to $2m$ terms.

(52.) Find the least multiple of 7, which divided by 2, 3, 4, 5, 6, leaves always unity for remainder.

(53.) Find the relations which exist among the quantities m, n, p, q, when $mx^3 + nx^2 + px + q$ is a complete cube.

(54.) At an election, where every voter may vote for any number of candidates not greater than the number to be elected, there are 4 candidates and 3 members to be chosen. In how many ways may a man vote?

(55.) Solve the following equations:—

$$\sqrt{\frac{x}{y}} + \sqrt{\frac{y}{x}} = \frac{19}{\sqrt{xy}} - 1,$$

$$\sqrt[4]{xy} = \frac{5\sqrt{6}}{\sqrt{x} + \sqrt{y}}.$$

(56.) Prove that terms of an arithmetical progression, taken at equal intervals, form an arithmetical progression.

(57.) Solve the equation—

$$\sqrt{(x + \sqrt{x})} - \sqrt{(x - \sqrt{x})} = a\left(\frac{x}{x + \sqrt{x}}\right)^{\frac{1}{2}}.$$

(58.) Find the sum of n terms of the series—

$$1 + 3 + 6 + 10 + \ldots\ldots$$

(59.) Find the value of

$$\frac{\sqrt{(a^2 + x^2)} + x}{\sqrt{(a^2 + x^2)} - x} \text{ when } x = \frac{a(b - c)}{2\sqrt{bc}}.$$

(60.) A guard contains ten men and three officers. The patrol consists of three men and an officer. How many different patrols are there? In how many of these will

a specified man serve? a specified officer? a specified man with a specified officer?

(61.) Find the values for x, y, and z from the following equations:—

$$xz = y^2,$$
$$x + y + z = 14,$$
$$x^2 + y^2 + z^2 = 84.$$

(62.) A dealer takes £1000 to market to buy horses and cows. He finds that he can get five cows and a horse for £98, and five horses and a cow for £178. Required the prices of a horse and of a cow; the number of ways in which he may disburse his money; and what he will obtain in each way.

(63.) If s_1, s_2, s_3 be the sum of n, $2n$, and $3n$ terms of a geometrical progression, then will $s_1^2 + s_2^2 = s_1(s_2 + s_3)$.

(64.) Solve the equation—

$$x^2 + x\sqrt[3]{xy^2} = 208,$$
$$y^2 + y\sqrt[3]{x^2y} = 1053.$$

(65.) If $x = \dfrac{-1+\sqrt{-3}}{2}$ find the value of $x^2 - 3x + 4\sqrt{x}$.

(66.) Solve the equations—

(I.) $4096^x = \dfrac{8}{64^x}$;

(II.) $\dfrac{4^x}{2^{x+y}} = 8$, $x = 3y$.

(67.) Extract the square root of

$$ab + c^2 + \sqrt{\{(a^2 - c^2)(b^2 - c^2)\}}.$$

(68.) Given $\dfrac{\sqrt{(a+x)}}{\sqrt{a} + \sqrt{(a+x)}} = \dfrac{\sqrt{(a-x)}}{\sqrt{a} - \sqrt{(a-x)}}$ to find x.

(69.) If a^2, b^2, c^2 be in arithmetical progression, then $\dfrac{1}{b+c}$, $\dfrac{1}{c+a}$, $\dfrac{1}{a+b}$ are in arithmetical progression.

(70.) A point moves with a speed which is different in different miles, but invariable in the same mile, and its

speed in any mile varies inversely as the number of miles travelled before it commences this mile. If the second mile be described in two hours, find the time occupied in describing the nth mile.

(71.) If the square root of the product of two quantities is rational, show that the square root of the quotient obtained by dividing one by the other is also rational.

(72.) Solve the following equation :—

$$\frac{a^2+ax+x^2}{a^2-ax+x^2}=\frac{a^2}{x^2}.$$

(73.) Show that

$$\frac{15}{\sqrt{10}+\sqrt{20}+\sqrt{40}-\sqrt{5}-\sqrt{80}}=\sqrt{5}(1+\sqrt{2}).$$

(74.) Prove that every term of the series 1, 2, 4, ... is greater by unity than the sum of all that precede it.

(75.) Find the value of $\frac{2(\sqrt{5}-\sqrt{3})}{\sqrt{5}+\sqrt{3}}$ to five places of decimals.

(76.) Extract the square roots of

$$\frac{11+6\sqrt{2}}{\sqrt{3}},\ \frac{7-4\sqrt{3}}{\sqrt{5}},\ \text{and}\ \frac{27}{\sqrt{5}}-7.$$

(77.) Solve the equation, $2^{x+1}+4^x=80$.

(78.) Find the sum of the geometrical series

$$\frac{\sqrt{3}}{\sqrt{3}+1}+\frac{\sqrt{3}}{\sqrt{3}+2}+\ldots \text{ to 20 terms.}$$

(79.) The increase in the number of male and female criminals is 1·8 per cent.; the decrease in the number of males alone is 4·6 per cent.; and the increase in the number of females is 9·8. Compare the number of male and female criminals respectively.

(80.) Find the values of x and y in the equations

$$4(x+y)=3xy,$$
$$x+y+x^2+y^2=26.$$

(81.) Show that $\frac{c+2a}{c-b}+\frac{c+2b}{c-a}=4$, >7, or >10, according as c is the arithmetical, geometrical, or harmonical mean between a and b.

(82.) Show that if $\frac{ay-bx}{c}=\frac{cx-az}{b}=\frac{bz-cy}{a}$, then $\frac{x}{a}=\frac{y}{b}=\frac{z}{c}$.

(83.) Solve the following equations:—

$$\left(\frac{a^2-x^2}{y^2-b^2}+\frac{y^2-b^2}{a^2-x^2}\right)^{\frac{1}{2}}+\left(\frac{a^2+x^2}{y^2+b^2}+\frac{y^2+b^2}{a^2+x^2}\right)^{\frac{1}{2}}=4,\ xy=ab.$$

(84.) Sum to n terms $\left(r-\frac{1}{r}\right)^2+\left(r^2-\frac{1}{r^2}\right)^2+\left(r^3-\frac{1}{r^3}\right)^2+\ldots$

(85.) The number of combinations of $n+1$ things, taken $n-1$ together, is 36. What is the number of permutations of n things?

(86.) Find values for x and y in the following equations:—

$$x^2+y^2=3xy,$$
$$x^5+y^5=2.$$

(87.) Find the sum of the infinite series—

$$ar+(a+ab)r^2+(a+ab+ab^2)r^3+\ldots,$$

r and br being each less than unity.

(88.) Solve the equations—

$$x^2+y^2-(x+y)=a,$$
$$x^4+y^4+x+y-2(x^3+y^3)=b.$$

(89.) An annuity A is to commence at the end of p years and to continue q years; find the equivalent annuity to commence immediately and to continue q years.

(90.) Solve the equations—

$$x^2y^3z^4=a,\ x^3y^4z^2=b,\ x^4y^2z^3=c.$$

(91.) If $a+b+c=0$, $x+y+z+w=0$, then the two equations $\sqrt{(ax)}+\sqrt{(by)}+\sqrt{(cz)}=0$, $\sqrt{(bx)}-\sqrt{(ay)}+\sqrt{(cw)}=0$ are deducible the one from the other.

(92.) Find the sum of $1.3+4.5+7.7+10.9$, etc. to n terms.

(93.) Solve the following equation—

$$\sqrt[n]{(1+x)^2}-\sqrt[n]{(1-x)^2}=\sqrt[n]{(1-x^2)}.$$

(94.) A diamond of a carats is worth m times a ruby of b^2 carats, and both together are worth c pounds. Find the value of a diamond and of a ruby, each weighing w^2 carats, having given that the value of diamonds varies as the square of their weight, and the square of the value of rubies varies as the cube of their weight.

(95.) If y be the harmonical mean between x and z, and x and z respectively the arithmetical and geometrical means between a and b, prove that

$$y=\frac{2(a+b)}{\left\{\left(\frac{a}{b}\right)^{\frac{1}{4}}+\left(\frac{b}{a}\right)^{\frac{1}{4}}\right\}^2}.$$

(96.) Find values for x and y in the following:—

$$x^{x+y}=y^{4a},$$
$$y^{x+y}=x^a.$$

(97.) Show that $\left\{\frac{-1+\sqrt{(-3)}}{2}\right\}^n+\left\{\frac{-1-\sqrt{(-3)}}{2}\right\}^n$ is equal to 2 if n be a multiple of 3, and equal to -1 if n be any other integer.

(98.) Solve the equations

$$\frac{ax+by}{cz}=\frac{cz+ax}{by}=\frac{by+cz}{ax}=x+y+z.$$

(99.) Prove that

$$\text{Log}\left\{(1+x)^{\frac{1+x}{2}}(1-x)^{\frac{1-x}{2}}\right\}=\frac{x^2}{1.2}+\frac{x^4}{3.4}+\frac{x^6}{5.6}+\ldots$$

(100.) If $p_r=\frac{1.3.5\ldots.(2r-1)}{2.4.6\ldots.2r}$, prove that

$$p_{2n+1}+p_1p_{2n}+p_2p_{2n-1}+\ldots.+p_{n-1}p_{n+2}+p_np_{n+1}=\tfrac{1}{2}.$$

EXAMINATION PAPERS

FOR

SCIENCE SCHOOLS AND CLASSES.

Selected from those set during last ten years.

SECOND STAGE.

(1.) If $x=\frac{a+b}{c-d}$, then $(a-cx)^2+(x^2-1)(b^2-d^2)$ is a complete square.

(2.) Solve two of the following equations:—

(*a*) $\frac{15}{x-1}-\frac{8}{x-2}=\frac{3}{x-3}$;

(*b*) $\left(\frac{a^2+ax+x^2}{a^2-ax+x^2}\right)^2=\frac{a+2x}{a-2x}$;

(*c*) $\left.\begin{array}{l} x(y+z)=28 \\ y(x+z)=18 \\ z(x+y)=30 \end{array}\right\}$.

(3.) Show that if $\frac{x+y}{a+b}=\frac{y+z}{b+c}=\frac{z+x}{c+a}$, then $\frac{x}{a}=\frac{y}{b}=\frac{z}{c}$.

(4.) Solve two of the following sets of equations:—

(*a*) $\frac{x^2+5x+6}{x^2+x+1}=\frac{x^2+11x+30}{x^2+7x+5}$;

(*b*) $\left.\begin{array}{r} x+y+z=\ 1 \\ 3x+5y+7z=\ 11 \\ 9x+25y+49z=121 \end{array}\right\}$;

(*c*) $\left.\begin{array}{l} x^2+y=11x \\ y^2+x=11y \end{array}\right\}$.

(5.) Find the square root of $39+\sqrt{1496}$.

(6.) There are two numerical quantities such that twice the first added to the reciprocal of the second is equal to the second; while the

square of the second, together with the product of both, is equal to 4. Find them.

(7.) If $a+b:a-b::x+y:x-y$, show that $x^2+a^2:x^2-a^2::y^2+b^2:y^2-b^2$.

(8.) Solve the equations $x^2+2yz=y^2+2xz=z^2+2xy=3a^2$.

(9.) A vessel makes two runs on a measured mile, one with tide, in a minutes, and the other against tide, in b minutes. Find the speed of the vessel (through the water) and of the tide, supposing both to be uniform.

(10.) Show that the arithmetical, harmonical, and geometrical means of a and c are given by the value of b in the following proportions :—

$$a:a::a-b:b-c$$
$$a:b::a-b:b-c$$
$$a:c::a-b:b-c,$$

and state which is which.

(11.) Show that if
$$p^2-y^2=(x-a)^2 \quad [1]$$
$$q^2-y^2=(x-b)^2 \quad [2]$$
$$r^2-y^2=(x-c)^2 \quad [3]$$

and a, b, c are all different, then

$$p^2(b-c)+q^2(c-a)+r^2(a-b)+(b-c)(c-a)(a-b)=0.$$

(12.) Extract the square root of

$$4x^8+9x^6-12x^4+16x^2+9-2x(6x^6-8x^4+9x^2-12).$$

(13.) Find the value of $\frac{x}{y}$, having given $\frac{x^{2n}-ay^{2n}}{x^{2n}+ay^{2n}}=\frac{x^n-b(x-y)^n}{x^n+b(x-y)^n}$.

(14.) Show that $2(a-b)(a-c)+2(b-c)(b-a)+2(c-b)(c-a)$ is the sum of three squares.

(15.) Solve the equations—

$$xy+yz=15,$$
$$yz+zx=12,$$
$$zx+xy=7.$$

(16.) I have an alloy containing a ounces of tin, and b ounces of lead, and also another alloy containing c ounces of tin, and d ounces of lead. In what proportions must I mix them to get an alloy containing equal parts of each metal? What limitation is necessary to make this question possible, and how does this limitation appear in your algebraical work?

(17.) Calculate to four places of decimals, $\frac{4}{\sqrt{5}+1}$.

(18.) Simplify—

(a) $\frac{1}{x^2-5x+6}-\frac{3}{x^2-x-6}+\frac{2}{x^2+x-6}$;

(b) $\frac{x^3-\frac{3}{4}x+\frac{1}{4}}{(2x-1)^2}$.

(19.) Solve the following equations, giving all the values of x and y which satisfy them :—

$$(a)\ \left\{\begin{matrix} ax+by &= c \\ bx+ay &= d \end{matrix}\right\};$$

$$(b)\quad ax^2-bx = a+b;$$

$$(c)\ \left\{\begin{matrix} x^2+y &= 3 \\ x+y^2 &= 3 \end{matrix}\right\}.$$

(20.) A man buys a certain number of acres of land for £1600; by selling it at £28 an acre he gains as much as he gave for three acres. How many acres did he buy?

(21.) Given that $a:b::b:c$, to show that $a^2+b^2+c^2=(a+b+c)(a-b+c)$.

(22.) Show that the value of $(a^2-bc)+(b^2-ac)+(c^2-ab)$ is not altered if a, b, and c are each increased or diminished by the same quantity.

(23.) Find the values of x and y which satisfy the following equations :—

$$(a)\ \left\{\begin{matrix} x+\dfrac{2}{y} = 9 \\ 3x-\dfrac{1}{y} = 13 \end{matrix}\right\};$$

$$(b)\ m\left(x-\frac{1}{x}\right)+n\left(x+\frac{1}{x}\right)=0;$$

$$(c)\ \left\{\begin{matrix} 3x^2-7xy+2y^2 = 0 \\ 3x+4y = 12 \end{matrix}\right\}.$$

(24.) A carpet, in shape a rectangle, has an area of 315 square feet; if a piece is taken off so as to reduce its width by 3 feet, the remaining area is $\frac{1}{15}$ths of what it would have been had its length been reduced by 3 feet. What is its length and breadth?

(25.) Solve two of the following sets of equations :—

$$(a)\ \sqrt{(x^2+a^2)}-\frac{b^2}{\sqrt{(x^2+a^2)}}=\frac{b^2-a^2}{a};$$

$$(b)\ x+y=7,\ x^3+y^3=133;$$

$$(c)\ \left\{\begin{matrix} xy+\sqrt{x+y} = 11 \\ 2xy-\sqrt{x+y} = 13 \end{matrix}\right\}$$

(26.) Two towns on a uniformly flowing river are 27 miles apart. A steamboat takes an hour and a half on its downward trip from one town to the other, and a row-boat three hours. The steamboat returns against stream in one-tenth of the time that the row-boat takes. Required the velocity of the river and the speed of the boats in still water.

(27.) What is the least integral multiplier which will make $17x^5-68x^4y+102x^3y^2-68x^2y^3+17xy^4$ a complete cube?

(28.) Two boys start at the same instant from the same corner of a square, the length of one of whose sides is 200 yards, and they run round it in opposite directions: one (A) runs at the rate of 100 yards in 15 seconds, and loses two seconds in turning a corner; the other (B) runs at the rate of 100 yards in 16 seconds, and loses one second in turning a corner. Where do they meet?

(29.) Solve two of the following sets of equations, finding all the values of x, or x and y, which satisfy them :—

$$(a)\ \frac{1}{x+a}+\frac{1}{x+b}=\frac{1}{a-x}+\frac{1}{b-x};$$

$$(b)\ \left\{\begin{array}{l} x^2+y=51 \\ 2x^2+y^2=102 \end{array}\right\};$$

$$(c)\ \left\{\begin{array}{r} \sqrt{(x+y)}+\sqrt{(x-y)}=5 \\ \sqrt{(x^2-y^2)}=4\cdot 5 \end{array}\right\}.$$

(30.) There are two coins such that 15 of the first and 14 of the second have the same value as 35 of the first and 6 of the second. What is the ratio of the value of the first coin to that of the second?

(31.) A traveller starts from A towards B at 12 o'clock, and another starts at the same time from B towards A. They meet at 2 o'clock, at 24 miles from A, and the one arrives at A while the other is still 20 miles from B. What is the distance between A and B?

(32.) If $z=\sqrt{(x^2+y^2)}$, show that

$$x+y+z:-x+y+z::x-y+z:x+y-z.$$

THIRD STAGE.

(33.) Explain (*a*) why $a^{\frac{1}{3}}$ means the cube root of a; and simplify—

$$(b)\ \left(x^{\frac{1}{2}-\frac{1}{7}}\right)^{\frac{2}{3}}\left(x^{\frac{1}{7}-\frac{1}{3}}\right)^{\frac{2}{5}}\left(x^{\frac{1}{3}-\frac{1}{5}}\right)^{\frac{2}{7}},$$

$$(c)\ \left\{\sqrt{\frac{\sqrt{7}-\sqrt{5}}{2}}+\sqrt{\frac{\sqrt{7}+\sqrt{5}}{2}}\right\}^2.$$

(34.) If $x=\dfrac{2ab+b^2}{a^2+ab+b^2}$ and $y=\dfrac{a^2-b^2}{a^2+ab+b^2}$, show that $x^3+y=y^3+x$.

(35.) Solve the equation—

$$x^4+\tfrac{1}{4}=x\sqrt{2}\sqrt{x^4-\tfrac{1}{4}}.$$

(36.) Prove the Binomial Theorem for a positive integral index, and write down the expansions of $(1+x)^{-\frac{1}{2}}$ and of $(1-5x)^{\frac{17}{5}}$, each to four *terms*.

(37.) If $q = \frac{(1+r)^n - 1}{r(1+r)^n}$, show that $\frac{1}{1+r} = \frac{1}{q+1}\left\{q + \frac{1}{(1+r)^{n+1}}\right\}$.

(38.) Establish the formula for obtaining the present value of an immediate annuity for a given term of years, at a given rate of compound interest.

(39.) Solve the equations—

$$x^2 = \frac{39}{y} - \frac{14}{x}, \quad y^2 = \frac{42}{x} - \frac{13}{y}.$$

(40.) If $\left(\frac{1}{x} + \frac{2}{y} + \frac{1}{z}\right)^2 = \frac{(x+2y+z)^2}{xy^2z}$, show that either $x = z$ or $y^2 = xz$.

(41.) Show that $m(m-1)(m-3)(m-6)$ is divisible by 8, when m is an integer, either positive or negative.

(42.) Write down the general term of the expansion of $(1-x)^{-3}$ by the Binomial Theorem; and find the coefficient of x^n in the expansion of $\frac{1+x+x^2}{(1-x)^3}$.

(43.) In how many different ways can an even number of things ($2n$) be divided into two equal groups? Also, how many different pairs can be made of $2n$ different things?

(44.) The sum of n terms of a geometrical progression beginning with unity is p, and the difference between the $(n+1)$th term and the first term is q. Find the common ratio of the series.

(45.) Obtain x from the equations—

$$(1-x)^y = a(1+x)^{-y}, \quad (bx)^y = a.$$

(46.) Write down the first four terms of the expansion of $(2 - \frac{3}{5}x)^{-5}$.

(47.) Solve the equation—

$$\frac{x-3}{x-4} - \frac{x-4}{x-6} - \frac{x-21}{x-24} + \frac{x-16}{x-20} = 0.$$

(48.) The first term of a series is unity, and its nth term is $2^{n-1} + 2n - 2$. Find the sum of the first m terms, and verify your result by writing down the first six terms and adding them together.

(49.) Find the numerically greatest term in the binomial expansion of $(1+\frac{3}{4})^{\frac{24}{7}}$.

(50.) Establish the rule for finding the number of permutations among x things, including p of one sort, q of another, and so forth. Apply your rule to the letters of the word *Trinitarian*.

(51.) When can the limit of the value of an infinite geometrical series

be found? Find the value of n terms of $a+ar+ar^2+\ldots\ldots$, and the sum of the infinite series $\frac{2}{3}+\frac{1}{2}+\frac{3}{8}+\ldots\ldots$

(52.) Assuming the Binomial Theorem, show that when n is an integer, $(a+x)^n$ has $n+1$ terms, that the coefficient of every term is an integer, and that the coefficients of terms equidistant from either end are the same.

(53.) Find the middle term of $(1+x)^{2n}$, the xth term of $(3a-2x)^{\frac{1}{5}}$, and the greatest term of $(3+5x)^7$, when $x=\frac{1}{2}$.

(54.) Solve the equations—

$$x^2+y^2+z^2=r^2,$$
$$2x(y+z)=a^2-2yz=l^2+2yz.$$

(55.) What relations between the coefficients A, B, C, a, b, c, and k, will make $(A-k)x^2+(B-k)y^2+(C-k)z^2+2ayz+2bzx+2cxy$, a complete square, irrespectively of the values of x, y, and z?

(56.) The first and second terms of a progression are $5\frac{1}{3}$ and $2\frac{1}{2}$. Find the fourth term on the three suppositions that the progression is (a) arithmetical, (b) geometrical, (c) harmonical.

(57.) Solve $\dfrac{3^{5x}-2^{5x}}{3^{5x}+2^{5x}}=\dfrac{19}{35}$ and $\dfrac{\log 3x+\log 2x}{\log 3x-\log 2x}=\dfrac{5}{3}$.

(58.) Prove the rule for finding the number of combinations of m things taken n together, and show that it is equal to the number of combinations of m things taken $m-n$ together.

(59.) Write down the first four terms of the expansion of $(32-10x)^{-\frac{1}{5}}$.

(60.) Solve completely the equations—

$$x^2-y^2-z^2=0,$$
$$xz+y=19,$$
$$xy-z=17.$$

ANSWERS.

EX. XLVI.—(Page 240.)

(1.) $8a^3x^9$; $-64b^3c^6y^{12}$; $\frac{a^6c^9x^9y^6}{m^3n^3z^3}$; $-125x^{3n}$.

(2.) $a^{12}x^4$; $81a^4x^{-8}$; $\frac{16}{81}\cdot\frac{c^8}{y^{12}}$.

(3.) $a^{6m}y^{-6n}$; $\frac{64a^{24}}{x^{-6}}$; z^{18}.

(4.) $2^m a^{2m}$; $(-1)^m x^{m^2}y^{mn}$; $x^{m(p+q)}$.

(5.) $x^6+6x^5y+15x^4y^2+20x^3y^3+15x^2y^4+6xy^5+y^6$;

$a^5-10a^4x+40a^3x^2-80a^2x^3+80ax^4-32x^5$.

(6.) a^{mnp}; x^{-mnp}.

(7.) $1+16x+96x^2+256x^3+256x^4$.

(8.) $x^6-3x^5+9x^4-13x^3+18x^2-12x+8$.

(9.) $243a^{10}+2025a^8+6750a^6+11250a^4+9375a^2+3125$;

$243a^{10}-2025a^8+6750a^6-11250a^4+9375a^2-3125$.

(10.) $x^8+8ax^7+28a^2x^6+56a^3x^5+70a^4x^4+56a^5x^3+28a^6x^2+8a^7x+a^8$.

(11.) $2a^7+168a^5b^2+1120a^3b^4+896ab^6$.

(12.) $x^6-4x^5+10x^3-4x+1$.

(13.) $x^{12}+3x^{10}+6x^8+7x^6+6x^4+3x^2+1$.

(14.) $a^6-4a^5+\frac{20}{3}a^4-\frac{160}{27}a^3+\frac{80}{27}a^2-\frac{64}{81}a+\frac{64}{729}$.

(15.) $x^{12}-3x^9+3x^6-x^3$.

(16.) $2(ax-by+cz-1)$.

(17.) $x^3-6x^2+15x-20+15x^{-1}-6x^{-2}+x^{-3}$.

(18.) $-283\frac{1}{2}$; 137.

EX. XLVII.—(Page 243.)

(1.) $\pm 12x^3$.
(2.) $-5x^3y^3z$.
(3.) $\pm 3\dfrac{ax^3}{b^2y}$.
(4.) $\dfrac{2x^2y^3}{3az^4}$.

(5.) $a^2(x-y)$.
(6.) $\pm(a-b)(x^2-y^2)^2$.
(7.) xy^2; $\pm a^nb^{3n}$.
(8.) $\pm a^2x$; $-xy^3$.

(9.) $5^3(a-x)^2(b-x)^{-1}$.
(10.) $\pm(a^2-x^2)$.
(11.) $\pm a^{2n-1}$.
(12.) $\dfrac{x^n(b-y)^m}{(a-x)^pz^{q+1}}$.

EX. XLVIII.—(Page 247.)

(1.) $a-3x$.
(2.) $4x+1$.
(3.) $3a^2-2ab+b^2$.
(4.) $2x+4+\dfrac{5}{x}$.
(5.) $\dfrac{x^3}{3}-\dfrac{y^2z}{2}$.
(6.) $\dfrac{2}{3}x-\dfrac{3}{4}$.
(7.) $x^2-2ax+a^2$.
(8.) $x^2-\dfrac{1}{2}x+1-\dfrac{1}{2}x^{-1}$.

(9.) $\dfrac{ax}{y}-\dfrac{by}{z}+\dfrac{cz}{x}$.
(10.) $a^{2m}-a^mb+2b^2$.
(11.) $1-\dfrac{x}{2}-\dfrac{x^2}{8}-\dfrac{x^3}{16}$,
and $a+2\dfrac{x^2}{a}-2\dfrac{x^4}{a^3}+4\dfrac{x^6}{a^5}$.
(12.) $1-\dfrac{x}{2}+\dfrac{3}{8}x^2-\dfrac{5}{16}x^3+\dfrac{35}{128}x^4$,
and $1+\dfrac{x}{a}+\dfrac{1}{2}\left(\dfrac{x}{a}\right)^2+\dfrac{1}{2}\left(\dfrac{x}{a}\right)^3+\dfrac{3}{8}\left(\dfrac{x}{a}\right)^4$.

EX. XLIX.—(Page 251.)

(1.) $x+2$.
(2.) $4x^2-y$.
(3.) $3a-2b+c$.
(4.) $2x^2-xy+3y^2$.
(5.) x^3+x^2+x+1.
(6.) $a-2+\dfrac{1}{a}$.
(7.) $1+\dfrac{1}{2}xy-\dfrac{1}{3}x^2y^2$.

(8.) $5x^2-y^n$.
(9.) $2x^4-x^3+\dfrac{1}{2}x^2$.
(10.) $ax-b+cx^{-1}$.
(11.) $m-\dfrac{1}{3}n+\dfrac{2}{9}\cdot\dfrac{n^2}{m}-\dfrac{14}{81}\cdot\dfrac{n^3}{m^2}$.
(12.) $1-\dfrac{1}{3}x-\dfrac{1}{9}x^2-\dfrac{5}{81}x^3-\dfrac{10}{243}x^4$.

EX. L.—(Page 254.)

(1.) x^2+2x-4.

(2.) $a-\dfrac{1}{2}a^{-1}x^2-\dfrac{1}{8}a^{-3}x^4-\dfrac{1}{16}a^{-5}x^6$.

(3.) $1+\dfrac{x}{5}-2\left(\dfrac{x}{5}\right)^2+6\left(\dfrac{x}{5}\right)^3-21\left(\dfrac{x}{5}\right)^4$.

EX. LI.—(Page 257.)

(1.) ± 2.
(2.) ± 3.
(3.) ± 4.
(4.) ± 1.
(5.) ± 3.
(6.) $\pm\sqrt{5}$.
(7.) ± 2.
(8.) $\pm\sqrt{6\frac{1}{2}}$.
(9.) $\pm\sqrt{ab}$.
(10.) $\pm\sqrt{\frac{1}{3}}$.
(11.) $\pm\sqrt{-1}$.
(12.) $\pm\sqrt{\frac{m-2}{m+4}}$.

EX. LII.—(Page 262.)

(1.) 2 or 4.
(2.) 4 or -5.
(3.) $\frac{2}{3}$ or -3.
(4.) 6 or $-\frac{1}{6}$.
(5.) $3\frac{1}{2}$ or $-1\frac{1}{2}$.
(6.) -1 or $-\frac{4}{7}$.
(7.) 5 or -1.
(8.) 3 or $\frac{5}{3}$.
(9.) 5 or -3.
(10.) 0 or 2 or 4.
(11.) $\frac{a+b}{a}$ or $\frac{a-b}{b}$.
(12.) No rational root.

EX. LIII.—(Page 266.)

(1.) 5 or -1.
(2.) 5 or 1.
(3.) 7 or 2.
(4.) 9 or -4.
(5.) $3\pm\sqrt{5}$.
(6.) 8 or $-2\frac{1}{2}$.
(7.) $\frac{1}{4}(7\pm\sqrt{17})$.
(8.) $\frac{2}{3}$.
(9.) $\frac{1\pm\sqrt{-3}}{2}$.
(10.) $\frac{2}{5}$ or $\frac{2}{3}$.
(11.) -11 or -6.
(12.) $\frac{1}{a}$ or -1.
(13.) $\frac{-25\pm\sqrt{85}}{9}$.
(14.) $-\frac{1}{2}$ or -2.
(15.) $\frac{a-b}{b}$ or $\frac{b}{a+b}$.
(16.) 0 or $\frac{1}{8}(3\pm\sqrt{44})$.
(17.) $a+4$.
(18.) $\frac{-1}{a+b+c}\left\{ab+ac+bc\mp\sqrt{a^2b^2+a^2c^2+b^2c^2-abc(a+b+c)}\right\}$.

EX. LIV.—(Page 268.)

(1.) $\frac{3}{8}$ or -6.
(2.) $\frac{6}{13}$ or -3.
(3.) $\frac{3}{4}$ or $\frac{4}{5}$.
(4.) $\frac{1}{22}(5\pm\sqrt{69})$.
(5.) $\frac{1}{64}(51\pm\sqrt{-343})$.
(6.) $\frac{-ac\pm\sqrt{ac(ac+4b^2)}}{2ab}$.

EX. LV.—(Page 270.)

(1.) ± 2 or ± 5.
(2.) $\pm 1\frac{1}{2}$ or $\pm\sqrt{2}$.
(3.) 2 or -1 or $\frac{1}{2}(1\pm\sqrt{-11})$.
(4.) 3 or -4 or 1 or -2.
(5.) 9 or 1 (1 inapplicable).
(6.) $\frac{9}{25}$ or 4 (4 inapplicable).
(7.) 5 or 8 (8 inapplicable).

(8.) 13 or 3 (3 inapplicable).

(9.) $\frac{2}{3}$ or $-\frac{3}{4}$ or $\frac{-1\pm\sqrt{865}}{24}$.

(10.) 2 or $-\frac{1}{2}$ or $\frac{-1\pm\sqrt{17}}{4}$.

(11.) $\frac{3a\pm\sqrt{a^2+4}}{2}$ or $\frac{3a\pm\sqrt{9a^2+4}}{2}$.

(12.) $x^m = 1\pm\sqrt{2}$ or 1.

EX. LVI.—(Page 274.)

(1.) 13 and 5.
(2.) $8\frac{1}{2}$ miles.
(3.) 16 miles and 15 miles.
(4.) 6 and 5.
(5.) 30 feet.
(6.) 18 minutes.
(7.) 45 miles and 42 miles.
(8.) 132 yards and 110 yards.
(9.) $22\frac{1}{2}$ gallons.
(10.) 30 days; 20 days; 60 days.
(11.) 80 feet by 60 feet.
(12.) 50 miles.
(13.) 10 shillings; 8 shillings.
(14.) 160 members; 6 shillings.
(15.) 1524.
(16.) 4 o'clock or 6 o'clock.
(17.) 11 and 7.
(18.) Train, 1 hour; coach, 4 hours.
(19.) £630; £735.
(20.) £1, 10s.; £3.
(21.) $2\frac{1}{2}$ miles per hour.
(22.) 240.
(23.) 18 feet and 16 feet.
(24.) £616; £624.
(25.) £40.
(26.) 8d. per dozen.
(27.) $3\frac{1}{4}$ and 3 miles per hour.
(28.) 9 gallons.
(29.) 5 and $4\frac{4}{5}$ miles per hour.
(30.) 4550.
(31.) 6; 7; 8; 9.
(32.) 5 hours; 7 hours.
(33.) $16\frac{2}{3}$ per cent.
(34.) 16 oxen.
(35.) £275; £225.
(36.) 72 bees.

EX. LVII.—(Page 287.)

(1.) 7; 4.
(2.) 5 or 2; 2 or 5.
(3.) 3 or -6; 2 or -25.
(4.) 2 or $1\frac{1}{3}$; 1 or $2\frac{1}{3}$.
(5.) ± 4 or ± 1; 1 or 16.
(6.) ± 6 or ± 5; $\pm 1\frac{2}{3}$ or ± 2.
(7.) 2 or 1; 1 or 2.
(8.) ± 7; ± 3.
(9.) 20 or 10; 15 or 0.
(10.) ± 4 or $\pm\frac{2}{\sqrt{19}}$; ± 2 or $\mp\frac{14}{\sqrt{19}}$.
(11.) 6 or 4; 4 or 6.
(12.) 8 or -6; 6 or -8.
(13.) $\frac{1}{4}$ or $\frac{1}{3}$; $\frac{2}{3}$ or $\frac{1}{2}$.
(14.) 3 or 1; 1 or 3.
(15.) 5 or -3; 3 or -5.
(16.) 7 or 4; 4 or 7.
(17.) ± 2 or $\pm\frac{1}{2}$; $\pm\frac{1}{2}$ or ± 2.
(18.) 3 or $\frac{1}{3}$; $\frac{1}{3}$ or 3.
(19.) ± 6 or $\pm 2\sqrt{-65}$; 2 or -20.
(20.) 2 or $\frac{1}{3}$ or $\frac{-3\pm\sqrt{17}}{2}$;
1 or $-\frac{2}{3}$ or $\frac{-5\pm\sqrt{17}}{2}$.
(21.) 9 or 2 or -6 or -7;
2 or 9 or -7 or -6.
(22.) 4 or 1 or $\frac{1}{2}(5\pm\sqrt{-159})$;
1 or 4 or $\frac{1}{2}(5\mp\sqrt{-159})$.
(23.) $\pm\sqrt{\pm 3}$; $\pm\sqrt{\pm 2}$.

(24.) ± 6 or ± 5 or $\pm\frac{1}{2}(\sqrt{33}+\sqrt{89})$; ± 5 or ± 6 or $\pm\frac{1}{2}(\sqrt{33}-\sqrt{89})$.

(25.) $\frac{1}{2}a$; $\frac{1}{2}a$.

(26.) $\frac{1}{20}(3n\pm\sqrt{9n^2+40m})$; $\frac{1}{20}(11n\mp 3\sqrt{9n^2+40m})$.

(27.) $\frac{a}{b}$ or $\frac{a}{a+b}$; 1 or $\frac{a+b}{b}$.

(28.) $a\pm 1$ or $-\frac{1}{2}(2a+1\mp\sqrt{4a+5})$; $a\mp 1$ or $-\frac{1}{2}(2a+1\pm\sqrt{4a+5})$.

(29.) $\frac{b^3}{a}$ or a^2; $\frac{a}{b}$ or $\frac{b}{a}$.

(30.) a^2+b^2; ab; etc.

EX. LVIII.—(Page 291.)

(1.) $x=\pm 4$; $y=\pm 5$; $z=\pm 6$.

(2.) $x=\pm\sqrt{\frac{ab}{c}}$; $y=\pm\sqrt{\frac{bc}{a}}$; $z=\pm\sqrt{\frac{ac}{b}}$.

(3.) $x=\pm 1$; $y=\pm 2$; $z=\pm 5$.

(4.) $x=\pm\frac{a}{\sqrt{a+b+c}}$; $y=\pm\frac{b}{\sqrt{a+b+c}}$; $z=\pm\frac{c}{\sqrt{a+b+c}}$.

(5.) $x=\pm 9$; $y=\pm 6$; $z=\pm 4$.

(6.) $x=\pm 1$; $y=\pm 2$; $z=\pm 3$.

(7.) $x=a$ or $\frac{57}{22}a$; $y=\frac{1}{2}b$ or $\frac{22}{9}b$; $z=\frac{1}{3}c$ or $\frac{18}{19}c$.

(8.) $x=\pm 5$; $y=\pm 2$; $z=\pm 1$.

(9.) $x=\pm 4$ or $\mp\frac{13}{\sqrt{7}}$; $y=\pm 3$ or $\mp\frac{5}{\sqrt{7}}$; $z=\pm 1$ or $\pm\frac{11}{\sqrt{7}}$.

(10.) $x=5$ or 2 or $\frac{1}{2}(3\pm\sqrt{-31})$; $y=3$ or 7; $z=2$ or 5 or $\frac{1}{2}(3\mp\sqrt{-31})$.

(11.) $x=\pm\frac{1}{2}\{a-b+\sqrt{(a+b)^2-8b^2}\}$; $y=\pm\frac{1}{2}\{a-b-\sqrt{(a+b)^2-8b^2}\}$; $z=\pm b$.

(12.) $x=\pm 8$ or $\pm\frac{7}{\sqrt{31}}$; $y=\pm 6$ or $\mp\frac{27}{\sqrt{31}}$; $z=\pm 3$ or $\pm\frac{51}{\sqrt{31}}$.

EX. LIX.—(Page 294.)

(1.) 18 passengers; 6d.

(2.) Tea, 3s.; coffee, 1s. 8d.

(3.) 9 feet; 13 feet.

(4.) 220 yards, and 176 yards per minute; 1980 yards.

(5.) 10 hours; 15 hours.

(6.) 3 miles; 11 miles; 34 miles.

(7.) $\frac{3}{4}$.

(8.) 32 yards; 30 yards; 24 yards.

(9.) A, 10 days; B, 12 days; C, 15 days.

(10.) Side of square, 12 feet; rectangle, 14 feet by 9 feet.

(11.) A, $3\frac{1}{2}$; B, 4; C, $2\frac{1}{2}$ miles per hour.

(12.) $\sqrt{ab}$ pounds; $10\sqrt{\left(\sqrt{\frac{a}{b}}-1\right)}$ years.

(13.) 343 cubic inches; 64 cubic inches.

(14.) Either 10s. per yard for the one and 5s. for the other, or 12s. 6d. per yard for the one and 2s. 6d. for the other.

(15.) A, 10 miles; B, 12 miles per hour.

(16.) 5 and 3.

(17.) £6000 at 7 per cent.; £7000 at 6 per cent.

(18.) $4\frac{1}{2}$ miles per hour walking; $4\frac{1}{2}$ miles per hour rowing.

(19.) 18 bought; 3 reserved.

(20.) 11 and 8.

(21.) 3 miles per hour.

(22.) 5 years at 4 per cent., or 8 years at $2\frac{1}{2}$ per cent.

(23.) 3 and 2.

(24.) 30 sheep at £1, 13s. 4d., and 10 cows at £5.

EX. LX.—(Page 307.)

(1.) $a^{\frac{1}{3}}$; $a^{\frac{2}{3}}x^{\frac{1}{3}}$; $2^{\frac{1}{3}}b^{\frac{1}{3}}x^{\frac{1}{3}}y^{\frac{1}{3}}$; $(a^2-x^2)^{\frac{1}{4}}x^{\frac{2}{4}}y^{\frac{2}{4}}$; $\dfrac{a^{\frac{1}{2}}b^{\frac{1}{2}}c^{\frac{1}{2}}}{x^{1\frac{1}{2}}y^{\frac{1}{2}}z^{1\frac{1}{2}}}$; $\dfrac{(a-b)^{\frac{2}{3}}}{(a+b)^{\frac{2}{3}}}x^{\frac{1}{3}}$.

(2.) $\sqrt[3]{a^2}$; $\sqrt[4]{a^3x}$; $\sqrt[6]{(a^2-1)x^2y^3}$; $\sqrt[8]{\dfrac{3(x-y)^3}{(x^2+y^2)^5}}$.

(3.) $\sqrt[3]{8a^9m^6(m-a)^3}$.

(4.) $\sqrt[6]{\dfrac{a^{16}x^{12}(a^2-x^2)^3}{729m^6n^2}}$.

(5.) $\sqrt{25xy}$; $-\sqrt[3]{8a^3(a^2+b^2)}$ or $\sqrt[3]{-8a^3(a^2+b^2)}$; $\sqrt[4]{\dfrac{a^3}{b^3}}$;

$\sqrt[5]{\dfrac{1}{9}(a^2-b^2)^2}$; $\sqrt[6]{a^2b^2(x^2-xy+y^2)}$.

(6.) $4\sqrt{7}$; $3a\sqrt[3]{3a^2b^2}$; $2(a-x)\sqrt[4]{4(a+x)^2}$ or $2(a-x)\sqrt{2(a+x)}$;

$(a-x)\sqrt{5a}$; $\dfrac{b^2x}{ay^2}\sqrt[6]{\dfrac{x^3}{y^2}}$; $\left(\dfrac{a^2-x^2}{x}\right)\sqrt[3]{\dfrac{1}{2}}$.

(7.) $4\sqrt[3]{\dfrac{1}{16}ax^3}$; $(a-x)\sqrt{\dfrac{a^2+ax+x^2}{a-x}}$.

(8.) $\sqrt[4]{9(a+x)^2(b+y)^2}$; $\{a^6(x-y)^4(y-z)^2\}^{\frac{1}{8}}$.

(9.) $\sqrt[6]{512}$; $\sqrt[6]{81}$; $(a^3b^4)^{1\frac{1}{2}}$; $\{a^6(a-x)^4\}^{1\frac{1}{2}}$; $\{a^6(b-x)^6\}^{1\frac{1}{2}}$.

(10.) $(a-x)\sqrt{ax(a+x)}$; $\dfrac{ay}{bz}\sqrt[3]{x^2}$.

(11.) $\sqrt[n]{a^{2n+1}x^{n+2}}$; $\left(\dfrac{ab}{a-b}\right)^{\frac{n-p}{n}}$.

(12.) $(a+x)\sqrt[4]{(a-x)^3}$; $ax\sqrt[n]{a^mx^m}$.

EX. LXI.—(Page 309)

(1.) $7\sqrt{2a}$.

(2.) $(3x+2y-xy)\sqrt[3]{3x}$.

(3.) $2a\sqrt{a+x}$.

(4.) $2\sqrt[4]{6xy}$.

(5.) $a\sqrt{3ax}$.

(6.) 0.

EX. LXII.—(Page 312.)

(1.) $x\sqrt{ab}$; $x\sqrt[6]{a^3b^4x}$.

(2.) $4\sqrt[4]{63}$; $24\sqrt[12]{4a^{11}}$.

(3.) $(a-b)\sqrt[12]{(a^2+ab+b^2)^6(a+b)^4(a-b)}$.

(4.) $\frac{1}{2}\sqrt{\frac{a}{b}}$; $\frac{3x}{4}\sqrt{\frac{}{y}}$.

(5.) $\frac{a^{\frac{3}{2}}x^{\frac{3}{2}}}{y^{\frac{1}{4}}}$; $\left(\frac{n}{m}\right)^{\frac{1}{6}}x^{\frac{1}{12}}$.

(6.) $a^{\frac{7}{8}}$; $x(x-y)^{\frac{1}{4}}$; $a^{\frac{mn}{m^2-n^2}}$;

$$\left\{\frac{a^m x^n}{(a+x)^{m+n}}\right\}^{\frac{1}{mn}}.$$

EX. LXIII.—(Page 314.)

(1.) $6\sqrt{a}-4\sqrt[3]{b}+2\sqrt[4]{c}-\sqrt[4]{2c}$.

(2.) $3\sqrt{xy}-2(2a-b)x+5\sqrt{x}+3\sqrt{y}$.

(3.) $a-b$.

(4.) $a+b$.

(5.) $a-b$.

(6.) a^2+ab+b^2.

(7.) $x+2\sqrt{xy}+y-xy$.

(8.) $x^{\frac{3}{5}}-2x^{\frac{1}{5}}+1$.

(9.) $m^{\frac{1}{2}}+3m^{\frac{1}{4}}p^{-\frac{1}{2}}-2p^{-1}$.

(10.) $\frac{\sqrt{3}-1}{\sqrt{3}+1}$.

(11.) $ax^{-\frac{1}{3}}+a^{-\frac{1}{3}}x^{\frac{2}{3}}-a^{-\frac{2}{3}}x$.

(12.) $x^2(64-192x^{\frac{1}{2}}+240x^{\frac{1}{3}}$
$-160x^{\frac{1}{2}}+60x^{\frac{2}{3}}-12x^{\frac{5}{6}}+x)$.

EX. LXIV.—(Page 318.)

(1.) $\frac{2}{5}\sqrt{5}$; $\frac{3}{2}\sqrt[3]{2}$; $\frac{2}{3}\sqrt[4]{3}$.

(2.) $\sqrt{11}+3$; $\frac{1}{3}(\sqrt{10}-1)$.

(3.) $\frac{\sqrt[3]{a^2x}}{a^2x}$; $\frac{\sqrt[6]{a(a+x)^3(a-x)^5}}{a-x}$.

(4.) 0; $2-\sqrt{2}$.

(5.) $\frac{a^{\frac{3}{2}}-x^{\frac{3}{2}}}{a^2+ax+x^2}$; $\frac{a^{\frac{1}{4}}+x^{\frac{1}{4}}}{a-x}$.

(6.) $(\sqrt{a+1}-\sqrt{a})^2$.

EX. LXV.—(Page 322.)

(1.) $1-x^{\frac{1}{4}}+2x^{\frac{1}{2}}-3x^{\frac{3}{4}}$.

(2.) $2x+x^{\frac{1}{3}}-\frac{1}{2}x^{-\frac{1}{3}}$.

(3.) $\sqrt{3}+\sqrt{2}$; $\sqrt{3}-\sqrt{2}$; $2\sqrt{2}+\sqrt{5}$; $\sqrt{13}-2$.

(4.) $\frac{\sqrt{5}-\sqrt{3}}{\sqrt{2}}$; $\frac{3+\sqrt{5}}{\sqrt{2}}$; $\frac{\sqrt{13}+1}{2}$; $\frac{5-\sqrt{2}}{2}$.

(5.) $\sqrt{a}-\sqrt{a-1}$; $\frac{\sqrt{x+2y}+\sqrt{x-2y}}{\sqrt{2}}$.

(6.) $a-2-2\sqrt{2a}$; $\frac{\sqrt{4a^2-b^2}}{b}-1$.

EX. LXVI.—(Page 326.)

(1.) $a+b$; $\pm 2\sqrt{-1}$.
(2.) $-ab$; 9.
(3.) $(2+\sqrt{-1})(2-\sqrt{-1})$ or $(1+2\sqrt{-1})(1-2\sqrt{-1})$ etc.
(4.) $\frac{3}{4}\sqrt{3}$; $\sqrt{-1}$.
(5.) $1+\sqrt{-2}+\sqrt{-3}+\sqrt{-6}$ $-2\sqrt{3}-3\sqrt{2}$.
(6.) 1; 1.
(7.) $\frac{x^2-y^2}{x^2+y^2}$.
(8.) $a^3-3ab^2+(3a^2b-b^3)\sqrt{-1}$; 16.
(9.) $3-2\sqrt{-1}$; $2-3\sqrt{-1}$.
(10.) $\sqrt{-1}$; $4-a^2$.
(11.) $a^4-a^2b^2+16b^4$.
(12.) $a+\sqrt{-1}$.

EX LXVII.—(Page 330.)

(1.) 4.
(2.) $\frac{3}{7}$.
(3.) 9 or 0.
(4.) 4.
(5.) 8.
(6.) 4.
(7.) $\pm\frac{1}{4}\sqrt{5}$.
(8.) 5 or -3.
(9.) 3.
(10.) 1.
(11.) $a+b$.
(12.) 3 or $-1\frac{3}{8}$.
(13.) $\pm\frac{1}{2}\sqrt{3}$.
(14.) $\pm\sqrt{1+4(a-1)^2}$ or ± 1.
(15.) $\pm\sqrt{\frac{2}{3}}$ or ± 1.
(16.) 2 or -3.
(17.) 4 or $\frac{4}{81}$.
(18.) $\frac{b(\sqrt{a}-\sqrt{2})}{a\sqrt{2}-\sqrt{a}}$.
(19.) $x=16$ or 1; $y=1$ or 16.
(20.) $x=8$; $y=12\frac{1}{2}$.
(21.) $x=256$; $y=81$.
(22.) $x=(\sqrt{a}\pm\sqrt{b})^2$, $y=(\sqrt{a}\mp\sqrt{b})^2$.
(23.) $x=\pm 20\sqrt{-1}$, or 0.
(24.) $x=\left(\frac{a\pm b}{a\mp b}\right)^{\frac{2mn}{n-m}}$.

EX. LXVIII.—(Page 339.)

(1.) 23 : 48.
(2.) 9 : 10.
(3.) 16 : 15.
(4.) 27 : 125; 5 : 6.
(5.) $a+b : a-b$.
(6.) 9 : 10; 40 : 41.
(7.) 15; 18.
(8.) *See Art.* 109.
(9.) $ab(a+b)$.
(10.) $x=\pm 4$; $y=\pm 6$; $z=\pm 10$.
(11.) 5 parts of 1st to 6 parts of 2nd.
(12.) $x=(a-b)c^2$; $y=(c-a)b^2$; $z=(b-c)a^2$.

EX. LXIX.—(Page 346.)

(2.) 1 ; $2\sqrt{3}$.
(3.) 3.
(4.) 91.
(6.) $x = \sqrt{ab}$.
(7.) $(100+a)(b-c) : (100+b)(c-a)$.
(8.) (2), $(a-c)^2 > 0$ $\therefore (a+c)^2 > 4b^2$, or $a:b::b:c, a-b:b::b-c:c$, $a-b:b-c::b:c$, and since $b>c$, $\therefore a-b>b-c$.
(9.) Proof as in No. 8.
(10.) 14 : 15.
(11.) $x = 2y$ or $-\frac{1}{2}y$.

EX. LXX.—(Page 351.)

(1.) $x = 5y$, $\therefore x \propto y$.
(2.) 20.
(3.) $y = 3x^2 + \frac{4}{x}$.
(4.) 37.
(5.) $x^2+xy+y^2 = (m^2+m+1)y^2$.
(6.) $a^2+b^2 = \frac{2(m^2+1)}{m^2-1}\,ab$.
(7.) $\frac{ptr}{100} = i$, $t = \frac{i}{\frac{pr}{100}}$, $\therefore t \propto \frac{1}{\frac{pr}{100}}$.
(8.) $mn(x^2-y^2) = \frac{x}{y} \times \frac{y}{x} = 1$, or $x^2-y^2 = \frac{1}{mn}$.
(9.) $\frac{1}{2}\mathrm{BP} = \mathrm{A}$, $\therefore \mathrm{B} = \frac{2\mathrm{A}}{\mathrm{P}}$ or $\mathrm{B} \propto \frac{1}{\mathrm{P}}$.
(10.) 6 days.
(11.) 24 miles ; 71 carriages.
(12.) $s = \frac{1}{2}$ ft.^{2}.

EX. LXXI.—(Page 357.)

(1.) 38.
(2.) $8-4n$.
(3.) $-\frac{1}{16}$.
(4.) $\frac{1}{2}n(n+1)$; 500500.
(5.) 22 ; -180.
(6.) $\frac{n}{3}(n+2)$; 901.
(7.) $2n-1$; n^2.
(8.) 9.
(9.) $\frac{a-b}{m-n}$.
(10.) 58, 69, 80.
(11.) $1\frac{4}{6}$, $1\frac{1}{3}$, $\frac{4}{6}$, $\frac{1}{3}$, $-\frac{1}{6}$.
(12.) Art. 209.
(13.) 21.
(14.) 25 or 1.
(15.) Middle term $= a + \frac{n-1}{2}d$.
(16.) 5, 7, 9, etc.
(17.) 25 ; 25.
(18.) 240.
(19.) 21, 31, 41.
(20.) £278, 6s. 3d.
(21.) 9 P.M.
(22.) Common difference $= \frac{n}{2}(n-1)$.
(23.) 12 seconds ; 23 times.
(24.) 50 parties ; 60 days.

EX. LXXII.—(Page 363.)

(1.) $-(5n-2)(-1)^n$.
(2.) [I.] -20 ; [II.] 25 ; [III.] $-\frac{1}{2}\{1+(4n-1)(-1)^n\}$.

(3.) $-2n(2n+1)(-1)^n$.

(4.) $\frac{1}{8}\{1-(2n^2+4n+1)(-1)^n\}$.

(5.) 64; 59.

(6.) $\frac{1}{2}n(n+1)$; $\frac{1}{6}n(n+1)(n+2)$.

(7.) $\frac{1}{2}n(2n^2+3n-1)$.

(8.) $-r(4r+3)$; $3+r(4r+7)$; $\frac{1}{4}\{1-(4n^2+6n+1)(-1)^n\}$.

(9.) $\frac{1}{3}n(2n-1)(2n+1)$.

(10.) $\frac{1}{3}n(n+1)(6n^2+14n-5)$.

(11.) They are each equal to $\left\{\frac{n(n+1)}{2}\right\}^2$.

EX. LXXIII.—(Page 368.)

(1.) 31250; $\frac{64}{2187}$.

(2.) $\frac{2176}{2187}$.

(3.) $-\frac{1}{2}+\sqrt{-\frac{1}{2}}+1-\sqrt{-2}-2$.

(4.) 5461; 547.

(5.) $\frac{2441406}{1953125}$; $\frac{209715}{262144}$.

(6.) $1-(-2)^n$; $\frac{1}{4}\left(\frac{4^n-3^n}{3^{n-2}}\right)$.

(7.) $\frac{2^n-x^n}{x^{n-1}(2-x)}$; $\frac{x\{(2y)^n-(-x)^n\}}{2^{n-1}y^n(2y+x)}$

(8.) 36, 108, 324; $\frac{9}{4}$, $\frac{3}{2}$, 1, $\frac{2}{3}$, $\frac{4}{9}$.

(9.) 342; $a\sqrt{2}$.

(10.) 2; $\frac{1}{2}$; $1\frac{1}{2}$; $2\frac{3}{4}$.

(11.) $\frac{4}{9}$; $\frac{1}{11}$; $2\frac{13}{56}$; $-\frac{8}{15}$.

(12.) $\frac{5}{9}$; $\frac{23}{100}+\frac{\frac{45}{10000}}{1-\frac{1}{100}}=\frac{387}{1650}$.

(13.) 70153.

(14.) 18, 162.

(15.) $\frac{n\text{th term}}{\text{sum of following terms}}=ar^{n-1}\div\frac{ar^n}{1-r}=\frac{1-r}{r}$.

(16.) 16 gallons; 3 gallons, $6\frac{3}{8}$ pints.

(17.) $1+\frac{1}{2}+\frac{1}{2^2}+\text{etc.}$; $1+\frac{1}{6}+\frac{1}{6^2}+$ etc.

EX. LXXIV.—(Page 371.)

(1.) $(n-1)2^n+1$.

(2.) $\frac{1}{2}\{(4n-3)3^{n+1}+9\}$.

(3.) $\frac{1}{25}\{8-(15n+8)(-4)^n\}$.

(4.) $3-\frac{2n+3}{2^n}$; 3.

(5.) $\frac{4}{81}(10^{n+1}-10-9n)$.

(6.) $\frac{a}{1-mr^2}\left\{\frac{r^{2n}-1}{r^2-1}-\frac{m(m^n-1)}{m-1}r^{2n}\right\}$; $\frac{a}{(1-mr^2)(1-r^2)}$.

EX. LXXV.—(Page 375.)

(1.) $\frac{2}{11}, \frac{1}{4}, \frac{2}{5}$.

(2.) $\frac{3}{4}, \infty, -\frac{1}{2}$.

(3.) $1\frac{25}{78}, 1\frac{40}{51}, 3\frac{23}{26}$.

(4.) 3, 6, ∞, -6.

(5.) 6 and 24.

(6.) 36 and 4.

(7.) $\frac{1}{2}, \frac{1}{4}, \frac{1}{8}$.

(10.) 48 and 12.

(12.) $\frac{1}{a}+(q-p)d=\frac{1}{b}$,

$\frac{1}{b}+(r-q)d=\frac{1}{c}$, etc.

EX. LXXVI.—(Page 382.)

(1.) $x=1$ or 4; $y=3$ or 1.

(2.) $x=3$ or 31; $y=5$ or 2.

(3.) $x=34, 27, 20, 13$ or 6; $y=3, 7, 11, 15$ or 19.

(4.) $x=0, 7, 14, 21$ or 28; $y=48, 36, 24, 12$ or 0.

(5.) $x=20$; $y=4$.

(6.) $x=5$; $y=10$.

(7.) $x=13$; $y=5$.

(8.) $x=21$; $y=6$.

(9.) 5 ways.

(10.) 9 ways.

(11.) 11 ways.

(12.) None.

(13.) 33, 61, 89.

(14.) Either 12 at 6s. and 4 at 7s., or 5 at 6s. and 10 at 7s.

(15.) 8 ways.

(16.) $\frac{7}{12}$ and $\frac{4}{15}$ or $\frac{3}{12}$ and $\frac{9}{51}$.

(17.) 3 lbs. at 2s. 4d., 16 lbs. at 2s. 9d.; 3s. per lb.

(18.) Pay 17 crowns and receive 8 half guineas.

(19.) $x=12$, $y=10$, $z=16$.

(20.) Either 60 guineas, 30 crowns, and 10 florins, or 63 guineas, 11 crowns, and 26 florins.

(21.) 419.

(22.) 166 ways; 197, 702.

(23.) £120, £84, £28.

(24.) 2775.

EX. LXXVII.—(Page 389.)

(1.) 32760; 5814.

(2.) 5040.

(3.) 40320; 1680.

(4.) 15.

(5.) 13699.

(6.) 16.

(7.) $\frac{\lfloor 8}{\lfloor 2 \lfloor 3}$; $\frac{\lfloor 10}{\lfloor 2 \lfloor 2 \lfloor 2}$; $\frac{\lfloor 9}{\lfloor 3 \lfloor 3}$; $\frac{\lfloor 13}{\lfloor 2 \lfloor 2 \lfloor 3 \lfloor 4}$.

(8.) 5.

(9.) 4.

(10.) $\frac{\lfloor 12}{\lfloor 2 \lfloor 2 \lfloor 3 \lfloor 4}$.

(12.) 72.

EX. LXXVIII.—(Page 394.)

(1.) 1365; 969.

(2.) $\frac{21}{\lfloor 14 \lfloor 7}$; $\frac{100 \cdot 99 \cdot 98 \cdot 97}{1 \cdot 2 \cdot 3 \cdot 4}$.

(3.) $\frac{15 \cdot 14 \cdot 13 \cdot 12}{\lfloor 6}(110-96) = 637$.

(4.) 10 or 11.

(5.) 20 common years, 315 days, 20 hours.

(6.) 511.

(7.) $\frac{9 \cdot 8 \cdot 7}{1 \cdot 2 \cdot 3} = 84$.

(8.) 7.

(9.) 12.

(10.) 83; 42.

EX. LXXIX.—(Page 411.)

(1.) $3^9 - 9 \cdot 3^8x^2 + 36 \cdot 3^7x^4 - 84 \cdot 3^6x^6 + 126 \cdot 3^5x^8 - 126 \cdot 3^4x^{10} + 84 \cdot 3^3x^{12} - 36 \cdot 3^2x^{14} + 9 \cdot 3x^{16} - x^{18}$.

(2.) $16x^7$; $-112266a^6$.

(3.) $\frac{64}{729} - \frac{32}{81}x + \frac{20}{27}x^2 - \frac{20}{27}x^3 + \frac{5}{12}x^4 - \frac{1}{8}x^5 + \frac{1}{64}x^6$.

(4.) $252x^{10}$; $\frac{15}{2}$.

(5.) $1 - 9x + 54x^2 - 270x^3 + \text{etc.}$

(6.) x^{15}; $-16x^{30}$.

(7.) $1 + 22x + 198x^2 + 924x^3 + 2310x^4 + \text{etc.}$

(8.) The 3rd $= \frac{20}{9}$.

(9.) $1 - 2x - x^2 - \frac{4}{3}x^3 - \frac{7}{3}x^4 - \frac{14}{3}x^5 - \text{etc.}$

(10.) $\frac{9 \cdot 8 \ldots (11-r)}{1 \cdot 2 \ldots (r-1)}x^{r-1}$; $-\frac{1 \cdot 1 \cdot 3 \cdot 5 \ldots (2r-5)}{1 \cdot 2 \cdot 3 \ldots (r-1)}(2x)^{r-1}$.

(11.) $\frac{1}{a^{\frac{3}{2}}} - \frac{3x^2}{4a^{\frac{7}{2}}} + \frac{21x^4}{32a^{\frac{11}{2}}} - \frac{77x^6}{128a^{\frac{15}{2}}} + \frac{1155x^8}{2048a^{\frac{19}{2}}} - \text{etc.}$

(12.) The 8th and 9th when $x = 1$, the 5th when $x = \frac{1}{2}$.

(13.) $\frac{1}{a^{\frac{1}{2}}} + 3\frac{x^{\frac{1}{2}}}{a} + 5\frac{x}{a^{\frac{3}{2}}} + 7\frac{x^{\frac{3}{2}}}{a^2} + 9\frac{x^2}{a^{\frac{5}{2}}} + 11\frac{x^{\frac{5}{2}}}{a^3} + \text{etc.}$

(14.) $\frac{m(m+1) \ldots (m+r-2)}{\lfloor r-1}\left(\frac{x}{m}\right)^{r-1}$; $\frac{1 \cdot 6 \cdot 11 \ldots (5r-9)}{\lfloor r-1}(-x)^{r-1}$.

(15.) 3·0006854+ ; 1·9952623+.

(16.) After the 4th; after the 12th.

(17.) Put $\sqrt{2} = \frac{1}{\sqrt{\frac{1}{2}}} = \frac{1}{\sqrt{1-\frac{1}{2}}} = \left(1 - \frac{1}{2}\right)^{-\frac{1}{2}}$.

(18.) $-\frac{1 \cdot 3 \cdot 7 \cdot 11 \cdot 15}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5}\left(\frac{x}{2}\right)^5$.

(19.) $s_1+s_2=(a+x)^n$, and $s_1-s_2=(a-x)^n$, etc.

(20.) $\frac{1\cdot3\cdot5\ldots\ldots(2r-3)}{1\cdot2\cdot3\ldots\ldots(r-1)}\left(\frac{x}{3}\right)^{r-1}$; $s=1+\frac{1}{2}\cdot\frac{2x}{3}+\frac{1}{2}\cdot\frac{3}{2}\left(\frac{2x}{3}\right)^2+\text{etc.}$

$$=\left(1-\frac{2x}{3}\right)^{-\frac{1}{2}}=\left(\frac{1}{3}\right)^{-\frac{1}{2}}=\sqrt{3}.$$

(22.) Develop $\left(\frac{1+n}{1-n}\right)^n$ in the two forms $\left(1+\frac{2n}{1-n}\right)^n$ and $\left(1-\frac{2n}{1+n}\right)^{-n}$.

(23.) $1+n+\frac{n(n-1)}{1\cdot2}+\text{etc.}+n+2\frac{n(n-1)}{1\cdot2}+\text{etc.}=2^{n-1}(2+n)$.

(24.) See No. 6, page 410.

EX. LXXX.—(Page 425.)

(1.) 2; $2\frac{1}{2}$; 8.
(2.) 4; 2; 6.
(3.) 3; 1; 4.
(4.) $3\log a+2\log b-\log c$; $\frac{2}{3}\log a+\frac{1}{12}\log b-\frac{3}{4}\log c$.
(5.) $\frac{1}{3}(a+b)$.
(6.) $m+n$; $3m+n$; $m+3n$; $m-n$; $4m-n$.
(7.) ·6989700; 1·3979400; $\bar{1}$·0969100.
(8.) 1·6812413; 2·3521826; $\bar{3}$·7781513.
(9.) $\bar{1}$·3876400; $\bar{1}$·9043281; $\bar{2}$·5302716.
(10.) 2·5649493; 3·4339872.
(11.) 1·1139433; 1·4913617.
(12.) 2·0043214; 2·4099331.
(13.) 22; 15.
(14.) Solve the equations, $3\,\text{Log}\,2+2\,\text{Log}\,7=2\cdot5932860$, $6\,\text{Log}\,2+\text{Log}\,7=2\cdot6512780$; $\text{Log}\,686=2\cdot8363240$; $\text{Log}\cdot0875=2\cdot9420080$.

(15.) $\frac{2}{1+\text{Log}\,2}=1\cdot5372435$; $\frac{3\,\text{Log}\,5}{4\,\text{Log}\,2+2\,\text{Log}\,3}=\cdot4174877$.

(16.) (I.) $\pm\frac{a}{\sqrt{a-b}}$; $\pm\frac{b}{\sqrt{a-b}}$. (II.) $\left(\frac{\log a}{\log b}\right)^{\frac{b}{a-b}}$; $\left(\frac{\log a}{\log b}\right)^{\frac{a}{a-b}}$.

EX. LXXXI.—(Page 434.)

(1.) 7s. $5\frac{1}{4}$d. nearly.
(2.) £1309·725.
(3.) £540·452.
(4.) 39·516 years.

(5.) £12820.

(6.) £1648·721; $n = 20 \operatorname{Log} 2 = 6{\cdot}02$ years.

(7.) £29946.

(8.) £513·608.

(9.) 4 per cent.

(10.) $n = \dfrac{\operatorname{Log} 2}{\operatorname{Log} 1{\cdot}05} = 14{\cdot}2$ years.

(11.) The 1st, $\dfrac{\operatorname{Log} 2}{\operatorname{Log} R}$ years; the 2nd, $\dfrac{\operatorname{Log} 3}{\operatorname{Log} R}$ years.

(12.) The first by nearly £1, 1s. per annum.

MISCELLANEOUS EXAMPLES.—(Page 436.)

(1.) $\dfrac{x}{y} - 1 + \dfrac{y}{x}$.

(2.) $x^2 - 2x + 1$.

(3.) $x^{\frac{1}{2}}y^{-\frac{1}{2}}$ or $\left(\dfrac{x}{y}\right)^{\frac{1}{2}}$.

(4.) $x = 7$ or 11; $x = 18$ or -2.

(5.) $x = \pm\dfrac{4}{\sqrt{3}}$.

(6.) $2x^2 - y + y^2$.

(7.) x^{pq}.

(8.) $a^{\frac{1}{3}} + 2a^{\frac{2}{3}}$.

(9.) 16.

(10.) 1001 : 1000; 1501 : 1500.

(11.) $x = 10$.

(12.) The number of answers is unlimited, the least being $x = 29$ and $y = 20$.

(13.) $x^2 + 2x - 4$.

(15.) $1 + \left(\dfrac{n-1}{2}\right)^2$.

(16.) $1 - \sqrt{x} + x$.

(17.) $\dfrac{(a+b+c)}{a-b+c} = \dfrac{(a+b+c)^2}{(a+c+b)(a+c-b)}$ etc.

(18.) $x = \frac{1}{2}$ or $\pm\sqrt{3}$.

(19.) $\dfrac{x(y \pm \sqrt{my})}{y-m}$.

(20.) $3 - 2\sqrt{-1}$; 6.

(21.) 19 oxen, 1 sheep, 80 hens.

(22.) $m = n^{\frac{1}{n-1}}$.

(23.) $x = \frac{1}{20}$ or $\frac{1}{11}$; $x = 16$ or -20.

(24.) $x = \pm 9$; $y = \pm 3$.

(25.) $-\frac{1}{2}$, $-\frac{1}{11}$, 0, $\frac{1}{11}$, $\frac{1}{2}$.

(26.) $1 + \sqrt{2}$; $1 + \sqrt{2} + \sqrt{3}$.

(27.) $\dfrac{5}{\sqrt{2}(\sqrt{5}-1)}$ or $\frac{1}{2}(\sqrt{10} + \sqrt{2})$.

(28.) $x = 3$ or $-1\frac{1}{11}$; $y = 4$ or $-2\frac{4}{11}$.

(29.) 2s.; 4s.; 5s.

(30.) $\dfrac{\sqrt{13} + \sqrt{-1}}{\sqrt{2}}$.

(31.) Omit the great bell, 5040.

(32.) 440 yards and 352 yards per minute.

(33.) $A = mn^{\frac{1}{2}}c^{\frac{3}{2}}$; $\therefore A \propto C^{\frac{3}{2}}$.

(34.) $c = \dfrac{b^2}{3a^3}$; $d = \dfrac{b^3}{27a^6}$.

(35.) $x = \pm a\sqrt{\left(\dfrac{1 \pm \sqrt{5}}{2}\right)}$.

(36.) 10100 yards, or $5\frac{14}{15}$ miles.

(37.) First prove $mc^2 = ny^2$.

(38.) $\dfrac{\sqrt{a} + \sqrt{b}}{\sqrt[4]{ab}}$; $a - b$.

(39.) $\frac{12.11.10}{1.2.3} \cdot \frac{8.7}{1.2} = 6160.$

(40.) No. (See Art. 226.)

(41.) Divide by $1+x$, and x^2,
$x = \frac{3 \pm \sqrt{5}}{2}$ or $\frac{-3 \pm \sqrt{-55}}{8}$.

(42.) $(a+c)^2 : (b+c)^2 = a : b$.

(43.) $p+q+(m-1)2q$.

(44.) $x = 6$ or 3; $y = 3$ or 6.

(45.) $\sqrt{\frac{4a-b}{b}}$.

(46.) 45 miles and 30 miles.

(47.) $\sqrt{1 \mp \frac{2}{\sqrt{5}}}$.

(48.) 4 or -3 or $\frac{1 \pm \sqrt{-43}}{2}$.

(50.) £212$\frac{16}{17}$; £159$\frac{7}{17}$; £127$\frac{11}{17}$.

(51.) $-m(8m+9)$.

(52.) 301.

(53.) $n^2 : p^2 :: mp : nq$, and
$n^3 : p^3 :: m : q$.

(54.) $\frac{4.3.2}{1.2.3} + \frac{4.3}{1.2} + \frac{4}{1} = 14$.

(55.) $x = 9$ or 4; $y = 4$ or 9.

(57.) $x = \left\{ \frac{(a-1)^2+1}{2(a-1)} \right\}^2$.

(58.) $\frac{n}{6}(n+1)(n+2)$.

(59.) $\frac{b}{c}$.

(60.) 360; 108; 120; 36.

(61.) $x = 8$ or 2; $y = 4$; $z = 2$ or 8.

(62.) Horse, £33; cow, £13. Two ways — either buying 11 horses and 49 cows, or 24 horses and 16 cows.

(64.) $x = \pm 8$; $y = \pm 27$.

(65.) 3.

(66.) $x = \frac{1}{6}$; $x = 4\frac{1}{2}$, $y = 1\frac{1}{2}$.

(67.) $\sqrt{\left\{ \frac{(a+c)(b+c)}{2} \right\}} + \sqrt{\left\{ \frac{(a-c)(b-c)}{2} \right\}}$.

(68.) $x = 0$ or $\pm \frac{\sqrt{3}}{2}a$.

(70.) $2(n-1)$ hours.

(72.) $x = \frac{a}{2}(-1 \pm \sqrt{5})$ or $\pm a\sqrt{-1}$.

(74.) Sum of n terms $= 2^n - 1$,
$(n+1)$th term $= 2^n$.

(75.) ·25403.

(76.) $\frac{3+\sqrt{2}}{\sqrt[4]{3}}$, $\frac{2-\sqrt{3}}{\sqrt[4]{5}}$, $\frac{7-\sqrt{5}}{\sqrt[4]{20}}$.

(77.) $x = 3$.

(78.) $\frac{\sqrt{3}}{\sqrt{3}-1}\{1-(\sqrt{3}-1)^{20}\}$.

(79.) Female criminals $= \frac{1}{4}$ of male

(80.) $x = 4$ or 2; $y = 2$ or 4.

(81.) Use $\frac{a^2+b^2}{ab} > 2$.

(82.) Put the fractions each $= m$, and show that $(a^2+b^2+c^2)m = 0$. As the *sum of three squares* cannot be nothing, then $ay = bx$, etc.

(83.) $x = \pm b\left(\frac{\sqrt{3} \pm 1}{\sqrt{2}}\right)$,
$y = \pm a\left(\frac{\sqrt{3} \mp 1}{\sqrt{2}}\right)$.

(84.) $\frac{r^{2n}-1}{r^2-1}\left(r^2 + \frac{1}{r^{2n}}\right) - 2n$.

(85.) 40320.

(86.) $x = \frac{\sqrt[5]{10}}{2\sqrt{5}}(\sqrt{5} \pm 1)$,

$y = \frac{\sqrt[5]{10}}{2\sqrt{5}}(\sqrt{5} \mp 1)$.

(87.) $\frac{ar}{(1-r)(1-br)}$.

(88.) Add, then $(x^2-x)^2+(y^2-y)^2 = a+b$, from first $(y^2-y)^2 = \{a-(x^2-x)\}^2$, $\therefore x = \frac{1}{2}\{1 \pm \sqrt{1+2a \pm 2\sqrt{2(a+b)-a^2}}\}$.

(89.) $\frac{A}{R^\nu}$.

(90.) $x = \sqrt[17]{\frac{bc^{10}}{a^8}}$, $y = \sqrt[17]{\frac{ab^{10}}{c^8}}$, $z = \sqrt[17]{\frac{ca^{10}}{b^8}}$.

(91.) $(a+b)(x+y) = c(z+w)$, etc.

(92.) $\frac{n}{2}(4n^2+5n-3)$.

(93.) $x = \frac{(1 \pm \sqrt{5})^n - 2^n}{(1 \pm \sqrt{5})^n + 2^n}$.

(94.) Value of diamond is $\frac{mcw^4}{(m+1)a^2}$ pounds; value of ruby is $\frac{cw^3}{(m+1)b^3}$ pounds.

(96.) $x = \frac{1}{2}(4a+1 \mp \sqrt{8a+1})$,

$y = \frac{1}{2}(-1 \pm \sqrt{8a+1})$.

(98.) Either $ax = by = cz = \frac{2abc}{ab+bc+ac}$, or $ax+by+cz = 0$ and $x+y+z = -1$.

EXAMINATION PAPERS.—(Page 445.)

(1.) $(ab+cd+b^2-d^2)^2$.

(2.) (*a*) 6 or $2\frac{1}{2}$; (*b*) $\pm a\sqrt{-2}$ or 0; (*c*) $x = 4$, $y = 2$, $z = 5$.

(3.) $\frac{x+y}{a+b} = \frac{y+z}{b+c} = \frac{z+x}{c+a} = \frac{x+y+z}{a+b+c}$; $\therefore \frac{x}{a} = \frac{y}{b} = \frac{z}{c}$.

(4.) (*a*) $x = 0$ or $-6\frac{1}{2}$; (*b*) $x = 3$, $y = -8$, $z = 6$; (*c*) $x = 10$ or $6 \pm 2\sqrt{6}$, $y = 10$ or $6 \mp 2\sqrt{6}$.

(5.) $\sqrt{22}+\sqrt{17}$.

(6.) $\pm\frac{1}{\sqrt{3}}$; $\pm\sqrt{3}$.

(7.) Prove first that $a : b :: x : y$.

(8.) $x = y = z = a$.

(9.) $\frac{30(b+a)}{ab}$, $\frac{30(b-a)}{ab}$ miles per hour.

(10.) 1st, Arithmetical; 2nd, Geometrical; 3rd, Harmonical.

(11.) Take [2] from [1], and [3] from [2], multiply the first remainder by $b-c$, and the second by $a-b$, and again subtract.

(12.) $2x^4-3x^3+4x+3$.

(13.) $\frac{x}{y} = \frac{b^{\frac{1}{n}} \pm \sqrt{b^{\frac{1}{n}}\left(b^{\frac{1}{n}}+4a^{\frac{1}{n}}\right)}}{2b^{\frac{1}{n}}}$.

(14.) $(a-b)^2+(b-c)^2+(c-a)^2$.

(15.) $x=\pm 1$, $y=\pm 5$, $z=\pm 2$.

(16.) $\frac{d-c}{d+c}$ ounces of the first to $\frac{a-b}{a+b}$ ounces of the second. If $a>b$ then $d>c$ as appears from the above numerators.

(17.) 1·2360.

(18.) (*a*) $\frac{12}{(x^2-4)(x^2-9)}$; (*b*) $\frac{x+1}{4}$.

(19.) (*a*) $x=\frac{ac-bd}{a^2-b^2}$, $y=\frac{ad-bc}{a^2-b^2}$; (*b*) $x=\frac{a+b}{a}$ or -1; (*c*) $x=2$ or -1 or $\frac{\pm\sqrt{13}-1}{2}$, $y=-1$ or 2 or $\frac{\pm\sqrt{13}-1}{2}$.

(20.) 60 acres.

(21.) $a^2+b^2+c^2=(a+c)^2-b^2$, etc.

(22.) Substitute $a\pm x$ for a, etc.

(23.) (*a*) $x=5$, $y=\frac{1}{2}$; (*b*) $x=\pm\sqrt{\frac{m-n}{m+n}}$; (*c*) $x=\frac{4}{5}$ or $\frac{12}{5}$, $y=\frac{12}{5}$ or $\frac{4}{5}$.

(24.) 21 feet by 15 feet.

(25.) (*a*) $x=\frac{\sqrt{b^4-a^4}}{a}$ or 0; (*b*) $x=5$ or 1, $y=1$ or 5; (*c*) $x=8$ or 1, $y=1$ or 8.

(26.) 14 miles, 5 miles, and 4 miles per hour.

(27.) $289x^2(x-y)^2$.

(28.) A goes $403\frac{7}{31}$ yards, B goes $396\frac{24}{31}$ yards.

(29.) (*a*) $x=\frac{1}{2}\sqrt{2(a^2+b^2)}$; (*b*) $x=\pm 7$ or $\pm\sqrt{51}$, $y=2$ or 0; (*c*) $x=8$, $y=\pm\frac{5}{2}\sqrt{7}$.

(30.) The ratio is 2 to 5.

(31.) 60 miles.

(32.) $x^2=z^2-y^2=(z+y)(z-y)$, etc.

(33.) (*a*) For $a^{\frac{1}{3}}.a^{\frac{1}{3}}.a^{\frac{1}{3}}=a^{\frac{1}{3}+\frac{1}{3}+\frac{1}{3}}=a$; (*b*) 1; (*c*) $\sqrt{7}+\sqrt{2}$.

(35.) $x=\pm\sqrt{\frac{1}{2}(1\pm\sqrt{2})}$.

(36.) Art. 237, $1-\frac{1}{2}x+\frac{3}{8}x^2-\frac{5}{16}x^3+$ etc., $1-17x+102x^2-238x^3+$ etc.

(38.) Arts. 264 and 265.

(39.) $x=4$, $y=2$.

(40.) Transpose and resolve into the factors $(x-z)^2(y^2-xz)=0$.

(41.) Put $m=2n$ or $2n+1$, then $m(m-1)(m-3)(m-6)=4n(n-3)(2n-1)(2n-3)$ or $4n(n-1)(2n+1)(2n-5)$. As either n or $n-1$ must be divisible by 2, and also either n or $n-3$, each of the expressions must be divisible by 8.

(42.) $\frac{r(r+1)}{2}x^{r-1}$; $\frac{3n^2+3n+2}{2}$.

(43.) $\frac{\lfloor 2n}{\lfloor n \rfloor^2}$; $\frac{2n(2n-1)}{1 \cdot 2}$.

(44.) $\frac{p+q}{q}$.

(45.) $x = \frac{-b \pm \sqrt{4+b^2}}{2}$.

(46.) $\frac{1}{2^2}\left(1+\frac{3}{2}x+\frac{27}{5 \cdot 2^2}x^2+\frac{189}{5^2 \cdot 2^3}+\text{etc.}\right)$.

(47.) $x = \frac{12}{23}$.

(48.) $S_m = 2^m - 1 + m(m^2-1)$; $S_6 = 1+4+8+14+24+42 = 93$.

(49.) The sixth term, $\frac{35 \cdot 23 \cdot 19 \cdot 17}{2^{18}}$.

(50.) Art. 230; $\frac{\lfloor 12}{5 \cdot \lfloor 8} = 415800$.

(51.) Art. 215; $\frac{a(r^n-1)}{r-1}$; $2\frac{2}{3}$.

(52.) Arts. 241, 242, 243.

(53.) $\frac{2n}{(\lfloor n)^2}x^n$; $\frac{1 \cdot 4 \cdot 9 \ldots (5x-11)}{\lfloor x-1}\frac{1}{(3a)^{x-\frac{2}{5}}}\left(\frac{2}{5}x\right)^{x-1}$;

fourth term, $7 \cdot 5^4 \cdot 3^4 x^3$.

(54.) $x = \frac{1}{2}(\sqrt{r^2+a^2}+\sqrt{r^2-l^2})$, $y+z = \frac{1}{2}\{2(r^2+a^2)+2(r^2-l^2)-4x^2\}^{\frac{1}{2}}$,

$y-z = \frac{1}{2}\{2(r^2-a^2)+2(r^2+l^2)-4x^2\}^{\frac{1}{2}}$.

(55.) $(A-k)(B-k)(C-k) = abc$, $A = \frac{bc}{a}+k$, $B = \frac{ac}{b}+k$, $C = \frac{ab}{c}+k$.

(56.) (*a*) $-3\frac{1}{6}$; (*b*) $\frac{1125}{2048}$; (*c*) $1\frac{7}{15}$.

(57.) $x = \frac{8}{5}$; $x = \frac{1}{4}\sqrt[3]{12}$.

(58.) Arts. 232, 233.

(59.) $\frac{1}{2}+\frac{x}{2^3}+\frac{3x^2}{2^6}+\frac{11x^3}{2^{13}}+$ etc.

(60.) $x = 5$ or -5 or $\pm\sqrt{-26}$, $y = 4$ or $-\frac{33}{13}$ or $\frac{\mp 17\sqrt{-26}-19}{25}$,

$z = 3$ or $-\frac{56}{13}$ or $\frac{\mp 19\sqrt{-26}+17}{25}$.